

THE CONFIGURATION SPACE AND PRINCIPLE OF VIRTUAL POWER FOR ROUGH BODIES

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ABSTRACT. In the setting of an n -dimensional Euclidean space, the duality between velocity fields on the class of admissible bodies and Cauchy fluxes is studied using tools from geometric measure theory. A generalized Cauchy flux theory is obtained for sets whose measure theoretic boundaries may be as irregular as flat $(n - 1)$ -chains. Initially, bodies are modeled as normal n -currents induced by sets of finite perimeter. A configuration space comprising Lipschitz embeddings induces virtual velocities given by locally Lipschitz mappings. A Cauchy flux is defined as a real valued function on the Cartesian product of $(n - 1)$ -currents and locally Lipschitz mappings. A version of Cauchy's postulates implies that a Cauchy flux may be uniquely extended to an n -tuple of flat $(n - 1)$ -cochains. Thus, the class of admissible bodies is extended to include flat n -chains and a generalized form of the principle of virtual power is presented. Wolfe's representation theorem for flat cochains enables the identification of stress as an n -tuple of flat $(n - 1)$ -forms.

1. INTRODUCTION

This work presents a framework for the formulation of some fundamental notions of continuum mechanics. Specifically, using elements from geometric measure and integration theory, we consider, within the geometric setting of \mathbb{R}^n , the class of admissible bodies, configurations of bodies in space, the configuration space, virtual velocities, and stress theory.

Cauchy's stress theorem is one of the central results in continuum mechanics. It asserts the existence of the stress tensor which determines the traction fields on the boundaries of the various bodies. As the traditional proof relies on locality and regularity assumptions, from both the validity and the applicability aspects, stress theory is closely associated with the proper choice of the class of bodies. Furthermore, an appropriate class of bodies should allow the formulation of the Gauss-Green theorem or a generalization thereof.

In light of these observations, formulations of the fundamentals of continuum mechanics have considered, since the middle of the 20th century, the appropriate choice of the class of bodies. In [22], Noll defined a body as a differentiable three-dimensional compact manifold with piecewise smooth boundary that can be covered by a single chart and is endowed with a measure space structure. The configurations of the body in space provide charts on the body manifold and a part of the body is defined as a compact subset of the body with piecewise smooth boundary. In [38, p. 466] Truesdell

and Toupin ignore the formal issue of admissible bodies and tacitly assume smoothness wherever necessary. Later on, in [37, p. 4], Truesdell adopted the structure suggested in Noll's [22]. The common ground for these early works is in the assumption that bodies in continuum mechanics should have a smooth structure so that the classical versions of the notions of mathematical analysis apply. In [22], it is shown that Cauchy's original postulate on the dependence of the traction on the exterior normal may be replaced by an additivity assumption on the system of forces and the principle of linear momentum.

Gurtin and Martins introduce in [10] the notion of a Cauchy flux in order to represent the collection of total forces applied to the collection of surface elements. A Cauchy flux is defined as an additive, area bounded set function acting on the collection of compatible surface elements of the body, and a weakly balanced Cauchy flux is defined as a volume bounded Cauchy flux.

It seems that [1] and [43] were the first to propose that the class admissible bodies in continuum physics should consist of sets of finite perimeter. In Ziemer's work, admissible bodies are defined as sets of finite perimeter and a weakly balanced Cauchy flux is shown to be represented by a measurable vector field. The works [11, 24], which followed, further extended these studies. In [11], the class of admissible bodies is defined as the class of normalized sets of finite perimeter while in [24], admissible bodies are defined as fit regions which are bounded regularly open sets of finite perimeter and of negligible boundary. These postulates enabled the authors to apply a version of the Green-Gauss theorem and consider sets that do not necessarily have smooth boundaries as bodies in continuum mechanics for which balance laws may be written.

In [30, 31], Silhavy considered bodies as sets of finite perimeter in a bounded open region of \mathbb{R}^n . The author employs a weak approach in the formulation of Cauchy's flux theorem. A weakly balanced Cauchy flux of class L^1 is shown to be represented by a Borel measurable vector field q of class L^1 with a divergence (in the sense of distributions) of class L^1 . Silhavy's approach gives rise to a Borel set N_0 of Lebesgue measure zero such that the flux vector field q represents the action of the Cauchy flux for any surface whose intersections with N_0 has Hausdorff area measure zero. The analysis presented in Silhavy's work allows for singularities in the flux vector field and provides for the first time the concept of almost every surface. In [31], formal definitions of the concepts of almost every body and almost every surface are given and a weakly balanced Cauchy flux of class L^p is represented by a measurable vector field of class L^p with a divergence of class L^p .

The notions of almost every body and almost every surface are examined in [2] and it is shown that the Cauchy flux is determined by its action on a collection of rectangular planar surfaces with edges parallel to the axes of \mathbb{R}^n . A similar extension for Cauchy interaction is presented in [20].

In [28], a weak formulation of p -grade continuum mechanics, for any integer $p \geq 1$, is presented in the setting of differential manifolds. Configurations are viewed as C^p -embeddings of the body manifold in the physical space and forces are viewed as elements of the cotangent bundle to the infinite dimensional configuration manifold of mappings. Forces are shown to be represented by measures on the p -th jet bundle. Such a measure serves as a generalization of the p -th order stress. The representation of forces by stress measures enables a natural restriction of forces to subbodies. The consistency conditions for a such a system of p -th order forces are examined in [29].

The term *fractal* was coined in 1975 by Mandelbrot to indicate a highly irregular geometric object (see [19]). Mandelbrot's seminal work was the beginning of a very large body of research concerning the fractal properties of various physical phenomena. A variety of approaches have been suggested for the adaptation of fractal objects to branches of mechanics, *e.g.*, [33, 34, 36, 35, 40, 41, 3, 25].

In [26, 27], Cauchy's flux theory is formulated using Whitney's geometric integration theory [39] and new developments by Harrison [12, 13, 14, 15]. Bodies are viewed as r -dimensional domains of integration in an n -dimensional Euclidean space with $r \leq n$. A body is identified as an r -chain, the limit of a sequence of polyhedral chains with respect to a norm which is induced by Cauchy's postulates. Three types of chains are examined: flat, sharp and natural chains, such that

$$\text{polyhedral} \subset \text{flat chains} \subset \text{sharp chains} \subset \text{natural chains}.$$

Flat $(n-1)$ -chains may represent the fractal boundaries of bodies and sharp chains are shown to represent even less regular $(n-1)$ -dimensional objects. Fluxes of a given extensive property are postulated to be $(n-1)$ -cochains, *i.e.*, elements of the dual to the Banach space of $(n-1)$ -chains. By the duality structure of Whitney's theory, as one allows for less regular domains of integration (chains), the resulting fluxes (cochains) become more regular, automatically.

Rough bodies, introduced by Silhavy [32], are sets whose measure theoretic boundaries are fractals in the sense that the outer normal is not defined almost everywhere with respect to the $(n-1)$ -Hausdorff measure.

The present work, describes a framework where the mechanics of bodies with fractal boundaries may be studied. Unlike [27], the point of view of geometric measure theory as in [4] is mainly adopted.

The universal body is modeled as an open subset of \mathbb{R}^n and bodies are modeled as flat chains. In addition to the properties of the class of admissible bodies, special attention is given to the study of the kinematics of such bodies in space. The appropriate class of admissible configurations appears to be the set of Lipschitz embeddings. This class enjoys two significant properties. Firstly, the set of Lipschitz embeddings of the universal body into space is an open subset of the locally convex topological vector space of all

Lipschitz mappings of the universal body into space equipped with the Whitney, or strong, topology. In addition, for Lipschitz mappings there is a well defined pushforward action on flat chains, such as those representing bodies. Therefore, the images of bodies under the pushforward action induced by a Lipschitz embedding preserve their structure and relevant properties (*e.g.*, the availability of a generalized Stokes theorem).

Adopting the point of view that virtual velocities are elements of the tangent bundle of the configuration manifold, as the configuration space is open in the space of Lipschitz mappings, virtual velocities may be identified with Lipschitz mappings of the universal body into space. Considering force and stress theory, it is noted that forces which are required only to be continuous linear functionals relative to the Lipschitz topology, as would be the analogue of [28], seem to be too irregular for the setting adopted here. In order to constitute a consistent force system which is represented by an integrable stress fields, balance and weak balance are postulated. It is shown further that balance and weak balance are equivalent together to continuity relative to the flat norm of chains.

The paper is constructed as follows. Sections 2–5 contain a short outline of the various notions of geometric measure theory which are used in this work. Section 2 reviews the notion of differential forms, currents, flat chains and cochains. Section 3 presents sets of finite perimeter as well as the corresponding definitions for bodies and material surfaces as currents. In Section 4 we discuss some of the properties of locally Lipschitz maps. In particular, the image of a flat chain under a Lipschitz mapping is examined. In addition, Lipschitz embeddings and the properties of the set they constitute are considered. This enables the presentation of a Lipschitz type configuration space in Section 6. In Section 5 we discuss the product of locally Lipschitz maps and flat chains. This multiplication operation is used in the definition of a local virtual velocity. Our main theorem is presented in Section 7 where we prove that a system of forces obeying balance and weak balance is equivalent to a unique n -tuple of flat $(n - 1)$ -cochains. Generalized bodies and surfaces are introduced in Sections 8. Virtual strains, or velocity gradients, stresses and a generalized form of the principle of virtual work are presented in Sections 9 and 10.

2. REVIEW OF ELEMENTS OF HOMOLOGICAL INTEGRATION THEORY

In this section, some of the fundamental concepts form the theory of currents in \mathbb{R}^n are presented. Throughout, the notation is mainly adopted from [4, Chapter 4]. The notion of flat forms needed for Wolfe’s representation theorem, originally presented in Whitney’s Geometric Integration Theory [39, Chapter VII], is formulated in this section by the tools of Federer’s Geometric Measure Theory.

Let U be an open set in \mathbb{R}^n and V a vector space. The notation $\mathfrak{D}^m(U, V)$ is used for the vector space of smooth, compactly supported V -valued differential m -forms defined on U and $\mathfrak{D}^m(U)$ is used as an abbreviation for $\mathfrak{D}^m(U, \mathbb{R})$. The notation $d\phi$ is used for the *exterior derivative* of $\phi \in \mathfrak{D}^m(U)$, an element of $\mathfrak{D}^{m+1}(U)$. The vector space $\mathfrak{D}^m(U)$ will be endowed with a locally convex topology induced by a family of semi-norms [4, p. 344] as in the theory of distributions.

A continuous linear functional $T : \mathfrak{D}^m(U) \rightarrow \mathbb{R}$ is referred to as an m -dimensional current in U . The collection of all m -dimensional currents defined on U forms the vector space $\mathfrak{D}_m(U)$ which is the vector space dual to $\mathfrak{D}^m(U)$. Let $T \in \mathfrak{D}_m(U)$ with $m \geq 1$ then ∂T , the *boundary* of T is the element of $\mathfrak{D}_{m-1}(U)$ defined by

$$\partial T(\phi) = T(d\phi), \quad \text{for all } \phi \in \mathfrak{D}^{m-1}(U). \quad (2.1)$$

The exterior derivative d is a continuous linear map $d : \mathfrak{D}^m(U) \rightarrow \mathfrak{D}^{m+1}(U)$. Thus, the boundary operation $\partial : \mathfrak{D}_{m+1}(U) \rightarrow \mathfrak{D}_m(U)$, viewed as the adjoint operator to d , is a continuous linear operator on currents.

As an example of a 0-current in U , let L^n , denote the n -dimensional Lebesgue measure in \mathbb{R}^n . Then, the restricted measure $L^n \llcorner U$ is the 0-current defined as

$$L^n \llcorner U(\phi) = \int_U \phi dL^n, \quad \text{for all } \phi \in \mathfrak{D}^0(U). \quad (2.2)$$

Given η , a Lebesgue integrable m -vector field defined on U , then, $L^n \wedge \eta$ denotes the m -current in U defined by

$$L^n \wedge \eta(\phi) = \int_U \phi(\eta) dL^n, \quad \text{for all } \phi \in \mathfrak{D}^m(U). \quad (2.3)$$

The inner product in \mathbb{R}^n induces an inner product in $\bigwedge_m \mathbb{R}^n$ and $|\xi|$ will denote the resulting norm of an m -vector ξ . Given $\phi \in \mathfrak{D}^m(U)$, for every $x \in U$, $\phi(x)$ is an m -covector, and we set

$$\|\phi(x)\| = \sup \{ \phi(x)(\xi) \mid |\xi| \leq 1, \xi \text{ is a simple } m\text{-vector} \}. \quad (2.4)$$

The *comass* of ϕ is defined by

$$M(\phi) = \sup_{x \in U} \|\phi(x)\|. \quad (2.5)$$

For $T \in \mathfrak{D}_m(U)$ the *mass* of T is dually defined by

$$M(T) = \sup \{ T(\phi) \mid \phi \in \mathfrak{D}^m(U), M(\phi) \leq 1 \}. \quad (2.6)$$

An m -dimensional current T is said to be *represented by integration* if there exists a Radon measure μ_T and an m -vector valued, μ_T -measurable function, \vec{T} , with $|\vec{T}(x)| = 1$ for μ_T -almost all $x \in U$, such that

$$T(\phi) = \int_U \phi(\vec{T}) d\mu_T, \quad \text{for all } \phi \in \mathfrak{D}^m(U). \quad (2.7)$$

A sufficient condition for an m -dimensional current, T , to be represented by integration is that T is a current of finite mass, *i.e.*, $M(T) < \infty$. An

m -current T is said to be *locally normal* if both T and ∂T are represented by integration and is said to be a *normal* current if it is locally normal and of compact support. The notion of normal currents leads to the definition

$$N(T) = M(T) + M(\partial T), \quad (2.8)$$

and clearly, every $T \in \mathfrak{D}_m(U)$ such that $N(T) < \infty$ is a normal current. The vector space of all m -dimensional normal currents in U is denoted by $N_m(U)$. For a compact set K of U , set

$$N_{m,K}(U) = N_m(U) \cap \{T \mid \text{spt}(T) \subset K\}. \quad (2.9)$$

For each compact subset K of U , define F_K , the K -flat semi-norm on $\mathfrak{D}^m(U)$, by

$$F_K(\phi) = \sup_{x \in K} \{\|\phi(x)\|, \|d\phi(x)\|\}. \quad (2.10)$$

Dually, the K -flat norm for currents $T \in \mathfrak{D}_m(U)$ is given by

$$F_K(T) = \sup \{T(\phi) \mid F_K(\phi) \leq 1\}. \quad (2.11)$$

Note that if $T \in \mathfrak{D}_m(U)$ such that $F_K(T) < \infty$, then, $\text{spt}(T) \subset K$. For a given compact subset $K \subset U$, the set $F_{m,K}(U)$ is defined as the F_K closure of $N_{m,K}(U)$ in $\mathfrak{D}_m(U)$. In addition, set

$$F_m(U) = \bigcup_K F_{m,K}(U), \quad (2.12)$$

where the union is taken over all compact subsets K of U . An element in $F_m(U)$ is referred to as a *flat m -chain in U* .

For $T \in \mathfrak{D}_m(U)$ with $\text{spt}(T) \subset K$ it can be shown that $F_K(T)$ is given by

$$F_K(T) = \inf \{M(T - \partial S) + M(S) \mid S \in \mathfrak{D}_{m+1}(U), \text{spt}(S) \subset K\}. \quad (2.13)$$

By taking $S = 0$ we note that

$$F_K(T) \leq M(T). \quad (2.14)$$

In addition, any element $T \in F_{m,K}(U)$ may be represented by $T = R + \partial S$ where $R \in \mathfrak{D}_m(U)$, $S \in \mathfrak{D}_{m+1}(U)$, such that $\text{spt}(R) \subset K$, $\text{spt}(S) \subset K$, and

$$F_K(T) = M(R) + M(S). \quad (2.15)$$

Flat chains have some desirable properties. We note that the boundary of a flat m -chain is a flat $(m-1)$ -chain. Moreover, as Section 4 will show, the flat topology is preserved under Lipschitz maps. From a geometric point of view the notion of a flat chain may be used to describe objects of irregular geometric nature such as the Sierpinski triangle. The following representation theorem reveals the measure theoretic regularity characterization of flat m -chains.

Theorem 1. [4, Section 4.1.18] *Let T be a flat m -chain in U with $\text{spt}(T) \subset K$. Then, for any $\delta > 0$ and $E = \{x \mid \text{dist}(K, x) \leq \delta\} \subset U$, the current T may be represented by*

$$T = L^n \wedge \eta + \partial(L^n \wedge \xi), \quad (2.16)$$

such that η is an $L^n \llcorner U$ -summable, m -vector field, ξ is a $L^n \llcorner U$ -summable $(m+1)$ -vector field and $\text{spt}(\eta) \cup \text{spt}(\xi) \subset E$.

A linear functional X defined on $F_m(U)$ such that there exists $0 < c < \infty$ with $X(T) \leq cF_K(T)$ for any compact $K \subset U$ and $T \in F_{m,K}(U)$, is referred to as a *flat m -cochain*. The flat norm of a cochains is given by

$$F(X) = \sup \{X(\mathcal{A}) \mid \mathcal{A} \in F_m(U), F_K(\mathcal{A}) \leq 1, K \subset U\}. \quad (2.17)$$

By Theorem 1, a dual representation for flat cochains is available by flat forms which we shall now introduce.

Given a differentiable mapping u defined on an open set of \mathbb{R}^n , its derivative will be denoted by Du and its partial derivative with respect to the j -th coordinate will be denoted by $D_j u$. For a smooth m -vector field η in U , the *divergence* $\text{div} \eta$ of η is an $(m-1)$ -vector field in U defined by

$$\text{div} \eta = \sum_{j=1}^n D_j \eta \lrcorner dx_j, \quad (2.18)$$

where dx_i , $i = 1, \dots, n$ denote the dual base vectors relative to the standard basis e_j , $j = 1, \dots, n$ in \mathbb{R}^n [4, Section 4.1.6]. For an integrable m -form ϕ in U , the *weak exterior derivative* of ϕ is defined as an $(m+1)$ form in U denoted by $\tilde{d}\phi$ and such that the equality

$$\int_U \tilde{d}\phi(\eta) dL^n = - \int_U \phi(\text{div} \eta) dL^n, \quad (2.19)$$

holds for all compactly supported, smooth $(m+1)$ -vector fields η on U . The weak exterior derivative is simply the exterior derivative taken in the distributional sense. Note that $\tilde{d}\phi$ is uniquely defined up to a set of $L^n \llcorner U$ -measure zero, thus, for $\phi \in \mathfrak{D}^m(U)$, the relation $\tilde{d}\phi = d\phi$ holds $L^n \llcorner U$ -almost everywhere.

Differential forms whose components are Lipschitz continuous are referred to as *sharp m -forms* (adopting Whitney's terminology [39, Section V.10]). By Rademacher's theorem, the exterior derivative for sharp m -forms exists $L^n \llcorner U$ -almost everywhere and the existence of the weak exterior derivative follows. Sharp forms are clearly a generalization of the notion of a smooth differential form and a further generalization is given by flat forms where the Lipschitz continuity is relaxed.

Definition 2. An m -form ϕ in U is said to be *flat* if

$$F(\phi) = \sup_{\eta, \xi} \left\{ \int_U (\phi(\eta) + \tilde{d}\phi(\xi)) dL^n \right\} < \infty, \quad (2.20)$$

where η and ξ are respectively m and $(m+1)$ compactly supported, $L^n \llcorner U$ -summable vector fields such that

$$\int_U (\|\xi\| + \|\eta\|) dL^n = 1. \quad (2.21)$$

It is further observed that for ϕ , a flat m -form in U ,

$$F(\phi) = \operatorname{ess\,sup}_{x \in U} \left\{ \|\phi(x)\|, \|\tilde{d}\phi(x)\| \right\}. \quad (2.22)$$

Alternative definitions for flat forms may be found in [39, Section IX.7] and [17, Section 5.5].

Remark 3. For ϕ , a flat m -form in U , and ω , a flat r -form in U , $\phi \wedge \omega$ is a flat $(m+r)$ -form in U . One may use the definition of the weak exterior derivative to show that

$$\tilde{d}(\phi \wedge \omega) = \tilde{d}\phi \wedge \omega + (-1)^m \phi \wedge \tilde{d}\omega. \quad (2.23)$$

The representation theorem of flat cochains is traditionally referred to as Wolfe's representation theorem, [39, Chapter IX], [4, Section 4.1.19]. It states that any flat m -cochain X in U is represented by a flat m -form denoted by D_X such that

$$X(L^n \wedge \eta + \partial(L^n \wedge \xi)) = \int_U [D_X(\eta) + \tilde{d}D_X(\xi)] dL^n, \quad (2.24)$$

for any η and ξ , compactly supported, $L^n \lrcorner U$ -summable m and $(m+1)$ -vector fields, respectively. It is further noted that the flat norm $F(X)$ for the cochain X is given by

$$F(X) = \operatorname{ess\,sup}_{x \in U} \left\{ \|D_X(x)\|, \|\tilde{d}D_X(x)\| \right\} \equiv F(D_X). \quad (2.25)$$

The *coboundary* of a flat m -cochain X is defined as the flat $(m+1)$ -cochain dX such that

$$dX(\mathcal{A}) = X(\partial\mathcal{A}), \quad \text{for all } \mathcal{A} \in F_m(U), \quad (2.26)$$

where it is noted that the same notation is used for the coboundary operator and the exterior derivative. The coboundary is the adjoint of the boundary operator and thus a continuous linear operator taking flat m -chains to flat $(m+1)$ -chains. It follows from the representation theorem of flat chains that the flat $(m+1)$ -cochain dX is represented by the flat $(m+1)$ -form $D_{dX} = \tilde{d}D_X$. The last equality is used as the definition of the exterior derivative of a flat form in [39, Section IX.12].

Given a flat m -cochain X in U and a flat r -cochain Y in U , then, $X \wedge Y$ is an $(m+r)$ -cochain represented by the flat $(m+r)$ -form $D_{X \wedge Y} = D_X \wedge D_Y$, and for a flat $(m+r)$ -chain $T = L^n \wedge \eta + \partial(L^n \wedge \xi)$, $X \wedge Y(T)$ is defined by Equation(2.24). Moreover, Equation (2.23) implies that

$$d(X \wedge Y) = dX \wedge Y + (-1)^m X \wedge dY. \quad (2.27)$$

For a flat m -cochain X and a flat r -chain T , such that $m \leq r$, the interior product $X \lrcorner T$ is defined as a flat $(r-m)$ -chain such that

$$X \lrcorner T(\omega) = (X \wedge \omega)(T), \quad \text{for all } \omega \in \mathfrak{D}^{r-m}(U), \quad (2.28)$$

where $X \wedge \omega$ is the flat r -cochain represented by the flat r -form $D_X \wedge \omega$.

3. SETS OF FINITE PERIMETER, BODIES AND MATERIAL SURFACES

In this section we lay down the basic assumptions regarding the material universe. Sets of finite perimeter, or Caccioppoli sets, will play a central role in the proposed framework. We first recall some of the properties of sets of finite perimeter. Extended presentations of the subject may be found in [9, 4, 43, 44].

Let U be a Borel set in an open subset of \mathbb{R}^n and $B(x, r)$ be the ball centered at $x \in \mathbb{R}^n$ with radius r . Define the U -density of the point x by

$$d(x, U) = \lim_{r \rightarrow 0} \frac{L^n(U \cap B(x, r))}{L^n(B(x, r))}, \quad (3.1)$$

where the limit exists. The measure theoretic boundary, $\Gamma(U)$, of the set U is defined by

$$\Gamma(U) = \{x \mid 0 < d(x, U) < 1\}. \quad (3.2)$$

Definition 4. A Borel set U in \mathbb{R}^n is said to be a *set of finite perimeter* if $L^n(U) < \infty$ and $H^{n-1}(\Gamma(U)) < \infty$, where $H^{n-1}(\Gamma(U))$ is the $(n-1)$ -Hausdorff measure of $\Gamma(U)$.

Definition 4 is adopted in [43] as the definition for the class of admissible bodies. Several equivalent definitions for a set of finite perimeter may be found in the literature. In [44, Section 5.4.1] a set of finite perimeter is viewed as a set whose characteristic function is a function of bounded variation. In [4, Section 4.5] a set of finite perimeter is viewed as a set which induces a locally integral current. In this work, Definition 4 is selected for its intuitive geometric interpretation. For a set of finite perimeter, the exterior normal $\nu(x)$ to U exists H^{n-1} -almost everywhere in $\Gamma(U)$ thus making a generalized version of the Gauss-Green theorem applicable.

At this point we adopt the framework of [30] for the class of admissible bodies and material surfaces. Let \mathcal{B} be an open set in \mathbb{R}^n . A *body* in \mathcal{B} is denoted by \mathcal{P} and is postulated to be a set of finite perimeter in \mathcal{B} . Stricly speaking, a set of finite perimeter is determined up to a set of L^n measure zero, thus as a point set, it is not uniquely defined. Formally, each set of finite perimeter determines an equivalence class of sets. A unique representation of a body is given by the identification of the body \mathcal{P} with $T_{\mathcal{P}}$, an n -current in \mathcal{B} defined as $T_{\mathcal{P}} = (L^n \llcorner \mathcal{P}) \wedge e_1 \wedge \cdots \wedge e_n$. By Equation (2.3),

$$T_{\mathcal{P}}(\omega) = \int_{\mathcal{P}} \omega(x) (e_1 \wedge \cdots \wedge e_n) dL_x^n, \quad \text{for all } \omega \in \mathfrak{D}^n(\mathcal{B}). \quad (3.3)$$

Using the terminology of currents represented by integration, $\mu_{T_{\mathcal{P}}} = L^n \llcorner \mathcal{P}$ and $\vec{T}_{\mathcal{P}} = e_1 \wedge \cdots \wedge e_n$ are the Radon measure and unit n -vector associated with the current $T_{\mathcal{P}}$.

Objects of dimension $(n-1)$ for which one can compute the flux will be referred to as material surfaces. Formally, a *material surface* is defined as a pair $\mathcal{S} = (\hat{\mathcal{S}}, v)$ where $\hat{\mathcal{S}}$ is a Borel subset of \mathcal{B} such that for some body \mathcal{P} we have $\hat{\mathcal{S}} \subset \Gamma(\mathcal{P})$ and v is the exterior normal of \mathcal{P} such that $v(x) = \nu_{\mathcal{P}}(x)$ is

defined H^{n-1} -almost everywhere on $\hat{\mathcal{S}}$. Let $v^*(x)$ be a the covector defined by

$$v^*(x)(u) = v(x) \cdot u, \quad \text{for all } u \in \mathbb{R}^n, \quad (3.4)$$

and set $\vec{T}_{\mathcal{S}}$ as the $(n-1)$ -vector

$$\vec{T}_{\mathcal{S}}(x) = v^*(x) \lrcorner e_1 \wedge \cdots \wedge e_n. \quad (3.5)$$

It is easy to show that $\vec{T}_{\mathcal{S}}(x)$ is a unit, simple $(n-1)$ -vector H^{n-1} -almost everywhere on $\hat{\mathcal{S}}$. We use $T_{\mathcal{S}}$ to denote the $(n-1)$ -current in \mathcal{B} induced by the material surface \mathcal{S} , such that $\mu_{T_{\mathcal{S}}} = H^{n-1} \llcorner \hat{\mathcal{S}}$ and $\vec{T}_{\mathcal{S}}(x)$ are the Radon measure and $(n-1)$ -vector associated with $T_{\mathcal{S}}$, and

$$T_{\mathcal{S}}(\omega) = \int_{\hat{\mathcal{S}}} \omega(x)(\vec{T}_{\mathcal{S}}(x)) dH_x^{n-1}, \quad \text{for all } \omega \in \mathfrak{D}^{n-1}(\mathcal{B}). \quad (3.6)$$

The unit $(n-1)$ -vector $\vec{T}_{\mathcal{S}}(x)$ is viewed as the natural $(n-1)$ -vector *tangent* to the material surface \mathcal{S} . By Equation (2.3) we may write

$$T_{\mathcal{S}} = \left(H^{n-1} \llcorner \hat{\mathcal{S}} \right) \wedge \vec{T}_{\mathcal{S}}. \quad (3.7)$$

Consider the material surface $\partial\mathcal{P} = (\Gamma(\mathcal{P}), \nu_{\mathcal{P}})$ naturally induced by the body \mathcal{P} . One has,

$$\begin{aligned} T_{\partial\mathcal{P}}(\omega) &= \int_{\Gamma(\mathcal{P})} \omega(x)(\vec{T}_{\partial\mathcal{P}}(x)) dH_x^{n-1}, \\ &= \int_{\Gamma(\mathcal{P})} (\omega(x) \lrcorner e_1 \wedge \cdots \wedge e_n) \cdot \nu_{\mathcal{P}}(x) dH_x^{n-1}, \\ &= \int_{\mathcal{P}} d\omega(x) (e_1 \wedge \cdots \wedge e_n) dL_x^n, \\ &= T_{\mathcal{P}}(d\omega), \\ &= \partial T_{\mathcal{P}}(\omega). \end{aligned} \quad (3.8)$$

where in the third line above Gauss-Green theorem [4, Section 4.5.6] was used. Thus, it is noted that $T_{\partial\mathcal{P}} = \partial T_{\mathcal{P}}$ as expected, and the material surface \mathcal{S} associated with the body \mathcal{P} may be written as

$$T_{\mathcal{S}} = (\partial T_{\mathcal{P}}) \llcorner \hat{\mathcal{S}}. \quad (3.9)$$

Since a Radon measure is a Borel regular measure, the current $\partial T_{\mathcal{P}} \llcorner \hat{\mathcal{S}}$ is well defined for any Borel set $\hat{\mathcal{S}}$ [4, p. 356].

For each $T_{\mathcal{P}}$, we observe that $M(T_{\mathcal{P}}) = L^n(\mathcal{P})$ and $M(\partial T_{\mathcal{P}}) = H^{n-1}(\Gamma(\mathcal{P}))$ correspond to the “volume” of the body and “area” of its boundary, respectively. By Equation (2.8) one has $N(T_{\mathcal{P}}) = L^n(\mathcal{P}) + H^{n-1}(\Gamma(\mathcal{P})) < \infty$, so that the current $T_{\mathcal{P}}$ is a normal n -current in \mathcal{B} . The open set \mathcal{B} is referred to as the *universal body* and we define the *class of admissible bodies*, $\Omega_{\mathcal{B}}$, as the collection of all bodies in the universal body \mathcal{B} , i.e.,

$$\Omega_{\mathcal{B}} = \{T_{\mathcal{P}} \mid \mathcal{P} \subset \mathcal{B}, T_{\mathcal{P}} = L^n \llcorner \mathcal{P} \in N_n(\mathcal{B})\}. \quad (3.10)$$

The result obtained in [11] implies that in case \mathcal{B} is assumed to be a set of finite perimeter, $\Omega_{\mathcal{B}}$ would have the structure of a Boolean algebra and would form a material universe in the sense of Noll [23]. In Section 8, a generalized class of admissible bodies will be defined for which a requirement that \mathcal{B} is a bounded set will be sufficient in order to construct a Boolean algebra structure.

The collection of all material surfaces in \mathcal{B} will be denoted by $\partial\Omega_{\mathcal{B}}$, so that

$$\partial\Omega_{\mathcal{B}} = \left\{ T_{\mathcal{S}} \mid T_{\mathcal{S}} = (\partial T_{\mathcal{P}}) \llcorner \hat{\mathcal{S}}, T_{\mathcal{P}} \in \Omega_{\mathcal{B}} \right\}. \quad (3.11)$$

By the definition of $T_{\mathcal{S}}$ it follows that $M(T_{\mathcal{S}}) = H^{n-1}(\hat{\mathcal{S}})$ for each $T_{\mathcal{S}} \in \partial\Omega_{\mathcal{B}}$. Thus $T_{\mathcal{S}}$ is a flat $(n-1)$ -chain of finite mass. The material surfaces $T_{\mathcal{S}}$ and $T_{\mathcal{S}'}$ are said to be *compatible* if there exists a body $T_{\mathcal{P}}$ such that $T_{\mathcal{S}} = (\partial T_{\mathcal{P}}) \llcorner \hat{\mathcal{S}}$ and $T_{\mathcal{S}'} = (\partial T_{\mathcal{P}}) \llcorner \hat{\mathcal{S}'}$. The material surfaces $T_{\mathcal{S}}$ and $T_{\mathcal{S}'}$ are said to be *disjoint* if $\text{clo}(\hat{\mathcal{S}}) \cap \text{clo}(\hat{\mathcal{S}'}) = \emptyset$.

4. LIPSCHITZ MAPPINGS AND LIPSCHITZ CHAINS

Lipschitz mappings will model configurations of bodies in space. In this section we review briefly some of their relevant properties.

A map $\mathcal{F} : U \rightarrow V$ from an open set $U \subset \mathbb{R}^n$ to an open set $V \subset \mathbb{R}^m$, is said to be a (*globally*) *Lipschitz map* if there exists a number $c < \infty$ such that $|\mathcal{F}(x) - \mathcal{F}(y)| \leq c|x - y|$ for all $x, y \in U$. The *Lipschitz constant* of \mathcal{F} is defined by

$$\mathfrak{L}_{\mathcal{F}} = \sup_{x, y \in U} \frac{|\mathcal{F}(y) - \mathcal{F}(x)|}{|y - x|}. \quad (4.1)$$

The map $\mathcal{F} : U \rightarrow V$ is said to be *locally Lipschitz* if for every $x \in U$ there is some neighborhood $U_x \subset U$ of x such that the restricted map $\mathcal{F}|_{U_x}$ is a Lipschitz map.

Let $\mathcal{F} : U \rightarrow \mathbb{R}^m$ be a locally Lipschitz map defined on the open set $U \subset \mathbb{R}^n$, then for every K , a compact subset of U , the restricted map $\mathcal{F}|_K$ is globally Lipschitz in the sense that $\mathfrak{L}_{\mathcal{F}, K}$, the K -Lipschitz constant of the map $\mathcal{F}|_K$, given by

$$\mathfrak{L}_{\mathcal{F}, K} = \sup_{x, y \in K} \frac{|\mathcal{F}(x) - \mathcal{F}(y)|}{|x - y|}, \quad (4.2)$$

is finite.

4.1. Differential topology of Lipschitz maps. The vector space of locally Lipschitz mappings from the open set $U \subset \mathbb{R}^n$ to the open set $V \subset \mathbb{R}^m$ is denoted by $\mathfrak{L}(U, V)$. For a compact subset $K \subset U$, define the semi-norm

$$\|\mathcal{F}\|_{\mathfrak{L}, K} = \max \{ \|\mathcal{F}|_K\|_{\infty}, \mathfrak{L}_{\mathcal{F}, K} \}, \quad (4.3)$$

on $\mathfrak{L}(U, V)$, where,

$$\|\mathcal{F}|_K\|_{\infty} = \sup_{x \in K} |\mathcal{F}(x)|. \quad (4.4)$$

The vector space $\mathfrak{L}(U, V)$ is endowed with the Lipschitz strong topology (see [7]). It is the analogue of Whitney's topology (strong topology) for the space of differentiable mappings between open sets (see [18, p. 35]) and is defined as follows. Given $\mathcal{F} \in \mathfrak{L}(U, V)$, for some indexing set Λ , let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open, locally finite cover of $U \subset \mathbb{R}^n$, $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ an open cover of $V \subset \mathbb{R}^m$ and $\mathcal{K} = \{K_\lambda\}_{\lambda \in \Lambda}$ a family of compact subsets in U such that $K_\lambda \subset U_\lambda$ and $\mathcal{F}(U_\lambda) \subset V_\lambda$ for all $\lambda \in \Lambda$. A neighborhood $B^\mathfrak{L}(\mathcal{F}, \mathcal{U}, \mathcal{V}, \delta, \mathcal{K})$ of \mathcal{F} in the strong topology is defined by $\mathcal{U}, \mathcal{V}, \mathcal{K}$ as above and a family of positive numbers, $\delta = \{\delta_\lambda\}_{\lambda \in \Lambda}$, as the collection of all $g \in \mathfrak{L}(U, V)$ such that $g(K_\lambda) \subset V_\lambda$ and $\|\mathcal{F} - g\|_{\mathfrak{L}, K_\lambda} < \delta_\lambda$, i.e.,

$$B^\mathfrak{L}(\mathcal{F}, \mathcal{U}, \mathcal{V}, \delta, \mathcal{K}) = \left\{ g \in \mathfrak{L}(U, V) \mid g(K_\lambda) \subset V_\lambda, \|\mathcal{F} - g\|_{\mathfrak{L}, K_\lambda} < \delta_\lambda \right\}. \quad (4.5)$$

Definition 5. A map $\varphi : U \rightarrow V$, with $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, open sets such that $m \geq n$, is said to be a *bi-Lipschitz* map if there are numbers $0 < c \leq d < \infty$, such that [16, p. 78]

$$c \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq d, \quad \text{for all } x, y \in U, x \neq y. \quad (4.6)$$

Setting $L = \max\{\frac{1}{c}, d\}$,

$$\frac{1}{L} \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq L, \quad \text{for all } x, y \in U, x \neq y, \quad (4.7)$$

and in such a case φ is said to be *L-bi-Lipschitz*.

The map $\mathcal{F} : U \rightarrow V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets such that $m \geq n$, is a *Lipschitz immersion* if for every $x \in U$ there is a neighborhood $U_x \subset U$ of x such that $\mathcal{F}|_{U_x}$ is a bi-Lipschitz map, i.e., there are $0 < c_x \leq d_x < \infty$, and

$$c_x \leq \frac{|\varphi(y) - \varphi(z)|}{|y - z|} \leq d_x, \quad \text{for all } y, z \in U_x, y \neq z. \quad (4.8)$$

Definition 6. A Lipschitz map $\varphi : U \rightarrow V$ is said to be a *Lipschitz embedding* if it is a Lipschitz immersion and a homeomorphism of U onto $\varphi(U)$.

The following theorem pertaining to the set of Lipschitz embeddings is given in [7] and its proof is analogous to the case of differentiable mappings as in [18, p. 36–38].

Theorem 7. *The set $\mathfrak{L}_{\text{Em}}(U, V)$ is open in $\mathfrak{L}(U, V)$ with respect to the Lipschitz strong topology*

4.2. Maps of currents induced by Lipschitz maps. Since our objective is to represent bodies as currents, and in particular, as flat chains, and since we wish to represent configurations as Lipschitz mappings, we exhibit in the following the basic properties of the images of currents and chains under Lipschitz mappings.

Let T be a current on U and for open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, let $\mathcal{F} : U \rightarrow V$ be a smooth map whose restriction to $\text{spt}(T)$ is a proper map. For any r -form ω on V , the map \mathcal{F} induces a form $\mathcal{F}^\#(\omega)$, the *pullback* of ω by \mathcal{F} , defined pointwise by

$$\left(\mathcal{F}^\#(\omega)(x) \right) (v_1 \wedge \cdots \wedge v_r) = (\omega(\mathcal{F}(x))) (D\mathcal{F}(v_1) \wedge \cdots \wedge D\mathcal{F}(v_r)), \quad (4.9)$$

for all $v_1, \dots, v_r \in \mathbb{R}^n$. It is observed that since \mathcal{F} is proper only on $\text{spt}(T)$, for a form ω with a compact support, $\text{spt}(\mathcal{F}^\#(\omega))$ need not be compact. However, for a real valued function ζ defined on U which is compactly supported and $\zeta(x) = 1$ for all x in a neighborhood of $\text{spt}(T) \cap \text{spt}(\mathcal{F}^\#(\omega))$, the smooth form $\zeta \mathcal{F}^\#(\omega)$ is of compact support. Thus, the *pushforward* $\mathcal{F}_\#(T)$ of T by \mathcal{F} may be defined as the current on V given by

$$\mathcal{F}_\#(T)(\omega) = T(\zeta \mathcal{F}^\#(\omega)), \quad \text{for all } \omega \in \mathfrak{D}^r(V), \quad (4.10)$$

for any ζ with the properties given above [8, Section 2.3]. The definition of $\mathcal{F}_\#(T)(\omega)$ is independent of ζ and thus will be omitted in the following. The pushforward operation satisfies

$$\partial \mathcal{F}_\#(T) = \mathcal{F}_\#(\partial T), \quad (4.11)$$

$$\text{spt}(\mathcal{F}_\#T) \subset \mathcal{F}\{\text{spt}(T)\}. \quad (4.12)$$

By a direct calculation one obtains that

$$M(\mathcal{F}_\#(T)) \leq \left(\sup_{x \in K} |D\mathcal{F}(x)| \right)^r M(T). \quad (4.13)$$

Applying Equation (2.8) it follows that

$$N(\mathcal{F}_\#(T)) \leq N(T) \sup \left\{ \left(\sup_{x \in K} |D\mathcal{F}(x)| \right)^r, \left(\sup_{x \in K} |D\mathcal{F}(x)| \right)^{r-1} \right\}, \quad (4.14)$$

and by (2.15),

$$F_{\mathcal{F}\{K\}}(\mathcal{F}_\#(T)) \leq F_K(T) \sup \left\{ \left(\sup_{x \in K} |D\mathcal{F}(x)| \right)^r, \left(\sup_{x \in K} |D\mathcal{F}(x)| \right)^{r+1} \right\}, \quad (4.15)$$

where $\mathcal{F}\{K\}$ is the image of the set K under the map \mathcal{F} .

In case $\mathcal{F} : U \rightarrow V$ is a locally Lipschitz map, the map $\mathcal{F}_\#$ cannot be defined as in the case of smooth maps. However, given any compact $K \subset U$, for $T \in F_{r,K}(U)$, one may define the current $\mathcal{F}_\#(T)$ as a weak limit. Let $\{\mathcal{F}_\tau\}$, $\tau \in \mathbb{R}^+$, be a family of smooth approximations of \mathcal{F} obtained by mollifiers [4, Section 4.1.2]. (It is observed that flat chains have compact supports so that it is not necessary to require that \mathcal{F} is proper.) Set

$$\mathcal{F}_\#T(\omega) = \lim_{\tau \rightarrow 0} \mathcal{F}_\tau \# T(\omega), \quad \text{for all } \omega \in \mathfrak{D}^r(V).$$

The sequence $\{\mathcal{F}_{\tau\#}(T)\}$ is a Cauchy sequence with respect to the flat norm so that the limit is well defined and one may write

$$\mathcal{F}_{\#}(T) = \lim_{\tau \rightarrow 0} \mathcal{F}_{\tau\#}(T). \quad (4.16)$$

As a result, the locally Lipschitz map $\mathcal{F} : U \rightarrow V$ induces a map of flat chains

$$\mathcal{F}_{\#} : F_r(U) \rightarrow F_r(V).$$

Properties (4.11) and (4.12) hold for the map $\mathcal{F}_{\#}$ induced by a locally Lipschitz map \mathcal{F} and

$$M(\mathcal{F}_{\#}(T)) \leq M(T) (\mathfrak{L}_{\mathcal{F}, \text{spt}(T)})^r. \quad (4.17)$$

It follows that for normal currents

$$\begin{aligned} \mathcal{F}_{\#}(T) &\in N_{r, \mathcal{F}(K)}(V), \quad \text{for all } T \in N_{r, K}(U), \\ N(\mathcal{F}_{\#}(T)) &\leq N(T) \sup \left\{ (\mathfrak{L}_{\mathcal{F}, \text{spt}(T)})^r, (\mathfrak{L}_{\mathcal{F}, \text{spt}(T)})^{r-1} \right\}, \end{aligned} \quad (4.18)$$

and for flat chains

$$\begin{aligned} \mathcal{F}_{\#}(T) &\in F_{r, \mathcal{F}\{K\}}(V), \quad \text{for all } T \in F_{r, K}(U), \\ F_{\mathcal{F}\{K\}}(\mathcal{F}_{\#}(T)) &\leq F_K(T) \sup \left\{ (\mathfrak{L}_{\mathcal{F}, \text{spt}(T)})^r, (\mathfrak{L}_{\mathcal{F}, \text{spt}(T)})^{r+1} \right\}. \end{aligned} \quad (4.19)$$

See [4, Section 4.1.14] and [8, Section 2.3] for an extended treatment.

In Whitney's theory, the Lipschitz image of a flat chain \mathcal{A} is defined as follows [39, Chapter X]. First, for $P = \text{spt}(\mathcal{A})$ consider a full sequence of simplicial subdivision $\{P_i\}$ such that P_{i+1} is a simplicial refinement of P_i . Next, let $\{\mathcal{F}_i\}$ be a sequence of piecewise affine approximations of the Lipschitz map \mathcal{F} such that $\mathcal{F}_i(v) = \mathcal{F}(v)$ for all vertices v in the simplicial complex P_i . The chain $\mathcal{F}_{\#}(\mathcal{A})$ is defined as the limit in the flat norm of

$$\mathcal{F}(\mathcal{A}) = \lim_{i \rightarrow \infty} \mathcal{F}_i(\mathcal{A}). \quad (4.20)$$

Although Whitney's definition of $\mathcal{F}_{\#}(\mathcal{A})$ differs from that of Federer, the resulting chains are equivalent.

For a locally Lipschitz map $\mathcal{F} : U \rightarrow V$, and a flat m -cochain X in V , let $\mathcal{F}^{\#}(X)$ be the flat r -cochain in U defined by the relation

$$\mathcal{F}^{\#}(X)(T) = X(\mathcal{F}_{\#}(T)), \quad \text{for all } T \in F_r(U). \quad (4.21)$$

The flat r -cochain $\mathcal{F}^{\#}(X)$ is represented by the flat r -form $\mathcal{F}^{\#}(D_X)$, the pullback of the flat r -form D_X representing X by the map \mathcal{F} . Note that it follows from Rademacher's theorem, [4, Section 3.1.6], that $D\mathcal{F}$ exists L^n -almost everywhere in U . This does not limit the validity of Equation (4.9), as a flat form is defined only L^n -almost everywhere.

Consider a locally Lipschitz map $\mathcal{F} : U \rightarrow V$ from an open set $U \subset \mathbb{R}^n$ to an open set $V \subset \mathbb{R}^m$. For a flat n -cochain X in V and a current T_B

induced by an L^n -summable set B in U , one has

$$\begin{aligned}
\mathcal{F}^\#(X)(T_B) &= \int_B \mathcal{F}^\# D_X dL^n, \\
&= \int_B D_X(\mathcal{F}(x)) (D\mathcal{F}(x)(e_1) \wedge \cdots \wedge D\mathcal{F}(x)(e_n)) dL_x^n, \\
&= \int_B D_X(\mathcal{F}(x)) (e_1 \wedge \cdots \wedge e_n) J_{\mathcal{F}}(x) dL_x^n, \\
&= \int_{\mathcal{F}\{B\}} \sum_{x \in \mathcal{F}^{-1}(y)} D_X(y) dH_y^n.
\end{aligned} \tag{4.22}$$

In the last equation the area formula for Lipschitz maps [8, Section 2.1.2] was applied and $J_{\mathcal{F}}(x)$ is the Jacobian determinant of \mathcal{F} at x . In case $\mathcal{F} : U \rightarrow V$ is injective with $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, we have

$$\mathcal{F}^\#(X)(T_B) = X(\mathcal{F}_\# T_B) = \int_{\mathcal{F}\{B\}} D_X(y) dL_y^n = X(T_{\mathcal{F}\{B\}}), \tag{4.23}$$

thus, $\mathcal{F}_\# T_B = T_{\mathcal{F}\{B\}}$. In particular, for a body \mathcal{P} , we note that

$$\mathcal{F}_\# T_{\mathcal{P}} = T_{\mathcal{F}\{\mathcal{P}\}}. \tag{4.24}$$

For the material surface $T_{\partial \mathcal{P}}$, Equation (4.11) gives

$$\mathcal{F}_\#(T_{\partial \mathcal{P}}) = \mathcal{F}_\#(\partial T_{\mathcal{P}}) = \partial \mathcal{F}_\#(T_{\mathcal{P}}) = \partial T_{\mathcal{F}\{\mathcal{P}\}},$$

and for a material surface $T_{\mathcal{S}}$ Equation (3.9) implies that

$$\mathcal{F}_\#(T_{\mathcal{S}}) = T_{\mathcal{F}\{\mathcal{S}\}}. \tag{4.25}$$

5. THE MULTIPLICATION OF SHARP FUNCTIONS AND FLAT CHAINS

A real valued field over a body \mathcal{P} will be represented below by the product of the current $T_{\mathcal{P}}$ and a *sharp function*—a real valued locally Lipschitz mapping. (The terminology is due to Whitney [39, Section V.4].) The space of sharp functions will be denoted by $\mathcal{L}_{\mathfrak{s}}(U)$.

A sharp function $\phi \in \mathcal{L}_{\mathfrak{s}}(U)$ defines a flat 0-cochain α_ϕ on U as follows. Let ξ be an $L^n \lrcorner U$ -measurable function compactly supported in U . Then, $L^n \wedge \xi$ is a 0-current of finite mass in U as defined in Equation (2.2). We set

$$\alpha_\phi(L^n \wedge \xi) = \int_U \phi(x) (\xi(x)) dL^n. \tag{5.1}$$

For a compactly supported $L^n \lrcorner U$ measurable 1-vector field η , $L^n \wedge \eta$ is a 1-current of finite mass in U defined in Equation (2.3). Using the existence of the weak exterior derivative $\tilde{d}\phi$, $L^n \lrcorner U$ -almost everywhere, we set

$$\alpha_\phi(\partial(L^n \wedge \eta)) = \int_U \tilde{d}\phi(\eta(x)) dL^n, \tag{5.2}$$

and obtain expressions analogous to Wolfe's representation theorem (Equation (2.24)). Let $\mathcal{A} \in F_0(U)$ be a flat 0-chain in U . Applying Theorem 1, \mathcal{A}

may be expressed as $\mathcal{A} = L^n \wedge \xi + \partial(L^n \wedge \eta)$ with ξ and η as defined above. Set

$$\alpha_\phi(\mathcal{A}) = \alpha_\phi(L^n \wedge \xi + \partial(L^n \wedge \eta)), \quad (5.3)$$

so that α_ϕ defines a continuous, linear function of flat 0-chains. Applying Equation (2.25) we obtain

$$F(\alpha_\phi) = \sup_{x \in U} \left\{ |\phi(x)|, |\tilde{d}\phi(x)| \right\}. \quad (5.4)$$

For $\mathcal{A} \in F_r(U)$ and $\phi \in \mathcal{L}_s(U)$, define the multiplication $\phi\mathcal{A}$ by $\phi\mathcal{A} = \alpha_\phi \lrcorner \mathcal{A}$ using the interior product as defined in Equation (2.28). That is,

$$\phi\mathcal{A}(\omega) = (\alpha_\phi \lrcorner \mathcal{A})(\omega) = (\alpha_\phi \wedge \omega)(\mathcal{A}), \quad \text{for all } \omega \in \mathfrak{D}^r(U), \quad (5.5)$$

where $\alpha_\phi \wedge \omega$ is the flat r -cochain represented by the flat r -form $\phi \wedge \omega$. Note that by Equation (5.5)

$$\text{spt}(\phi\mathcal{A}) \subset \text{spt}(\phi) \cap \text{spt}(\mathcal{A}). \quad (5.6)$$

For the boundary of $\phi\mathcal{A}$ we first note that

$$\partial(\phi\mathcal{A})(\omega) = \phi\mathcal{A}(d\omega) = (\alpha_\phi \wedge d\omega)(\mathcal{A}), \quad \text{for all } \omega \in \mathfrak{D}^{r-1}(U). \quad (5.7)$$

By Equation (2.27)

$$d(\alpha_\phi \wedge \omega) = (d\alpha_\phi) \wedge \omega + \alpha_\phi \wedge d\omega, \quad (5.8)$$

so that

$$\begin{aligned} \partial(\phi\mathcal{A})(\omega) &= (d(\alpha_\phi \wedge \omega) - (d\alpha_\phi) \wedge \omega)(\mathcal{A}), \\ &= (\phi\partial\mathcal{A} - d\alpha_\phi \lrcorner \mathcal{A})(\omega). \end{aligned} \quad (5.9)$$

Hence we can write

$$\partial(\phi\mathcal{A}) = \phi\partial\mathcal{A} - d\alpha_\phi \lrcorner \mathcal{A}. \quad (5.10)$$

Remark 8. The multiplication of sharp functions and chains was originally defined in [39, Section VII.1] using the notion of continuous chains which are r -vector field approximations of r -chains.

Proposition 9. *Given a sharp function ϕ , for $\mathcal{A} \in N_{r,K}(U)$*

$$N_{r,K}(\phi\mathcal{A}) \leq \left(\sup_{x \in K} |\phi(x)| + r \mathfrak{L}_{\phi,K} \right) N_{r,K}(\mathcal{A}), \quad (5.11)$$

and for $\mathcal{A} \in F_{r,K}(U)$ (see [39, p. 208])

$$F_{r,K}(\phi\mathcal{A}) \leq \left(\sup_{x \in K} |\phi(x)| + (r+1) \mathfrak{L}_{\phi,K} \right) F_{r,K}(\mathcal{A}). \quad (5.12)$$

Proof. For $\mathcal{A} \in N_{r,K}(U)$ we have

$$\begin{aligned}
M(\phi\mathcal{A}) &= \sup_{\omega \in \mathfrak{D}^r(U)} \frac{|\phi\mathcal{A}(\omega)|}{M(\omega)}, \\
&= \sup_{\omega \in \mathfrak{D}^r(U)} \frac{|(\alpha_\phi \wedge \omega)(\mathcal{A})|}{M(\omega)}, \\
&= \sup_{\omega \in \mathfrak{D}^r(U)} \frac{|\int_U (\phi(x)\omega(x)) (\vec{T}_{\mathcal{A}}(x)) d\mu_{\mathcal{A}}|}{M(\omega)}, \\
&\leq \sup_{\omega \in \mathfrak{D}^r(U)} \frac{\sup_{x \in K} \|(\phi(x)\omega(x))\| M(\mathcal{A})}{M(\omega)}, \\
&\leq \sup_{x \in K} |\phi(x)| M(\mathcal{A}),
\end{aligned} \tag{5.13}$$

where in the third line we used the representation by integration of \mathcal{A} and in the fourth line the term $\sup_{x \in K} |\phi(x)|$ was extracted since $\text{spt}(\mathcal{A}) \subset K$.

In order to examine the term $M(\partial(\phi\mathcal{A}))$, we first apply Equation (5.10)

$$M(\partial(\phi\mathcal{A})) \leq M(\phi\partial\mathcal{A}) + M(d\alpha_\phi \lrcorner \mathcal{A}). \tag{5.14}$$

For the first term on the right-hand side we have,

$$M(\phi\partial\mathcal{A}) = \sup_{\omega \in \mathfrak{D}^{r-1}(U)} \frac{|\alpha_\phi \wedge \omega(\partial\mathcal{A})|}{M(\omega)} \leq \left(\sup_{x \in K} |\phi(x)| \right) M(\mathcal{A}). \tag{5.15}$$

For the second term,

$$\begin{aligned}
M(d\alpha_\phi \lrcorner \mathcal{A}) &= \sup_{\omega \in \mathfrak{D}^{r-1}(U)} \frac{|\int_U d\alpha_\phi \wedge \omega (\vec{T}_{\mathcal{A}}) d\mu_{\mathcal{A}}|}{M(\omega)}, \\
&\leq \sup_{\omega \in \mathfrak{D}^{r-1}(U)} \frac{\sup_{x \in K} \|\tilde{d}\phi(x) \wedge \omega(x)\| M(\mathcal{A})}{M(\omega)}, \\
&\leq \sup_{\omega \in \mathfrak{D}^{r-1}(U)} \binom{r}{1} \frac{\sup_{x \in K} |\tilde{d}\phi(x)| M(\omega) M(\mathcal{A})}{M(\omega)}, \\
&= r \left(\sup_{x \in K} |\tilde{d}\phi(x)| \right) M(\mathcal{A}),
\end{aligned} \tag{5.16}$$

where in the third line we used the fact that for an l -form ω and a k -form ω'

$$M(\omega \wedge \omega') \leq \binom{l+k}{k} M(\omega) M(\omega'), \tag{5.17}$$

as is shown in [5].

One concludes that

$$\begin{aligned}
N(\phi\mathcal{A}) &= M(\phi\mathcal{A}) + M(\partial(\phi\mathcal{A})), \\
&\leq \sup_{x \in K} |\phi(x)| M(\mathcal{A}) + \sup_{x \in K} |\phi(x)| M(\partial\mathcal{A}) + r \mathfrak{L}_{\phi, K} M(\mathcal{A}), \\
&\leq \left(\sup_{x \in K} |\phi(x)| + r \mathfrak{L}_{\phi, K} \right) N(\mathcal{A}).
\end{aligned} \tag{5.18}$$

For a flat r -chain $\mathcal{A} \in F_{r, K}(U)$ we use the representation given in Equation (2.15) by $\mathcal{A} = R + \partial S$ so that $F_K(\mathcal{A}) = M(R) + M(S)$.

We first observe that

$$\begin{aligned}
M(d\alpha_\phi \lrcorner S) &= \sup_{\omega \in \mathfrak{D}^r(U)} \frac{d\alpha_\phi \lrcorner S(\omega)}{M(\omega)}, \\
&= \sup_{\omega \in \mathfrak{D}^r(U), \text{spt}(\omega) \subset K} \frac{(d\alpha_\phi \wedge \omega)(S)}{M(\omega)}, \\
&\leq \frac{M(S) M(d\alpha_\phi \wedge \omega)}{M(\omega)}, \\
&\leq \frac{M(S)}{M(\omega)} \binom{r+1}{r} M(\omega) \sup_{x \in K} |\tilde{d}\phi(x)|,
\end{aligned} \tag{5.19}$$

and conclude that

$$M(d\alpha_\phi \lrcorner S) \leq (r+1) \mathfrak{L}_{\phi, K} M(S). \tag{5.20}$$

Estimating $F_K(\phi\mathcal{A})$, one has

$$\begin{aligned}
F_K(\phi\mathcal{A}) &= F_K(\phi R + \phi \partial S), \\
&\leq F_K(\phi R) + F_K(\phi \partial S), \\
&\leq F_K(\phi R) + F_K(d\alpha_\phi \lrcorner S + \partial(\phi S)), \\
&\leq F_K(\phi R) + F_K(d\alpha_\phi \lrcorner S) + F_K(\partial(\phi S)), \\
&\leq F_K(\phi R) + F_K(d\alpha_\phi \lrcorner S) + F_K((\phi S)), \\
&\leq M(\phi R) + M(d\alpha_\phi \lrcorner S) + M(\phi S), \\
&\leq \sup_{x \in K} |\phi(x)| M(R) + (r+1) \mathfrak{L}_{\phi, K} M(S) + \sup_{x \in K} |\phi(x)| M(S), \\
&\leq \left\{ \sup_{x \in K} |\phi(x)| + (r+1) \mathfrak{L}_{\phi, K} \right\} (M(R) + M(S)), \\
&= \left\{ \sup_{x \in K} |\phi(x)| + (r+1) \mathfrak{L}_{\phi, K} \right\} F(\mathcal{A}),
\end{aligned} \tag{5.21}$$

where in the third line we used Equation (5.10), in the sixth line we used Equation (2.14), and in the seventh line we used Equation (5.20). \square

The vector space of sharp functions defined on U and valued in \mathbb{R}^m is identified as the space of m -tuples of real valued sharp functions defined on U i.e. $\mathcal{L}_5(U, \mathbb{R}^m) = [\mathcal{L}_5(U)]^m$. For $\phi \in \mathcal{L}_5(U, \mathbb{R}^m)$ and $\mathcal{A} \in F_{r,K}(U)$ the flat r -chain $\phi\mathcal{A}$ is viewed as an element of the vector space of $(F_{r,K}(U))^m$, i.e., an m -tuple of flat r -chains in U with $(\phi\mathcal{A})_i = \phi_i\mathcal{A}$.

6. CONFIGURATION SPACE AND VIRTUAL VELOCITIES

Traditionally, a configuration of a body \mathcal{P} is viewed as a mapping $\mathcal{P} \rightarrow \mathbb{R}^n$ which preserves the basic properties assigned to bodies and material surfaces. Guided by our initial definition of a body $T_{\mathcal{P}}$ as a current induced by \mathcal{P} , a set of finite perimeter in the open set \mathcal{B} , a configuration of the body \mathcal{P} is defined as a mapping $\kappa_{\mathcal{P}} \in \mathfrak{L}_{\text{Em}}(\mathcal{P}, \mathbb{R}^n)$. To distinguish it from a configuration of the universal body to be considered below, such an element, $\kappa_{\mathcal{P}}$, will be referred to as a *local configuration*. The choice of Lipschitz type configurations is a generalization of the traditional choice of C^1 -embeddings usually taken in continuum mechanics.

It is natural therefore to refer to $\mathcal{Q}_{\mathcal{P}} = \mathfrak{L}_{\text{Em}}(\mathcal{P}, \mathbb{R}^n)$ as the *configuration space of the body \mathcal{P}* . Since a body is a compact set, it follows from Theorem 7 that $\mathcal{Q}_{\mathcal{P}}$ is an open subset of the Banach space $\mathfrak{L}(\mathcal{P}, \mathbb{R}^n) \cong \mathfrak{L}(\kappa_{\mathcal{P}}\{\mathcal{P}\}, \mathbb{R}^n)$.

For $\mathcal{P}, \mathcal{P}' \in \Omega_{\mathcal{B}}$ the local configurations $\kappa_{\mathcal{P}}, \kappa_{\mathcal{P}'}$ are said to be compatible if

$$\kappa_{\mathcal{P}}|_{\mathcal{P} \cap \mathcal{P}'} = \kappa_{\mathcal{P}'}|_{\mathcal{P} \cap \mathcal{P}'} . \quad (6.1)$$

Note that the intersection of two sets of finite perimeter is a set of finite perimeter, thus, the restricted map may be viewed as the configuration of the body $\mathcal{P} \cap \mathcal{P}'$.

A *system of compatible configurations* κ , is a collection of compatible local configurations $\kappa = \{\kappa_{\mathcal{P}} \mid \mathcal{P} \in \Omega_{\mathcal{B}}\}$. Clearly, a system of compatible configuration is represented by a unique element of $\mathfrak{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n)$. An element $\kappa \in \mathfrak{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n)$ will be referred to as a *global configuration*, and the *global configuration space* \mathcal{Q} is the collection of all global configurations, i.e.,

$$\mathcal{Q} = \mathfrak{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n) . \quad (6.2)$$

We will view the configuration space as a trivial infinite dimensional differentiable manifold, specifically, a trivial manifold modeled on a locally convex topological vector space as in [21, Chapter 9].

It is noted, in particular, that a Lipschitz embedding is injective and the image of a set of finite perimeter in \mathcal{B} is a set of finite perimeter in \mathbb{R}^n . In addition, as Section 4 indicates, Lipschitz mappings are the natural morphism in the category of sets of finite perimeters and in the category of flat chains. Thus, an element $\kappa \in \mathcal{Q}$ preserves the structure of bodies and material surfaces as required. That is, every $\kappa \in \mathcal{Q}$ induces a map $\kappa_{\#}$ of flat chains. For any $T_{\mathcal{P}} \in \Omega_{\mathcal{B}}$, the current $\kappa_{\#}(T_{\mathcal{P}})$ is an element of $N_n(\mathbb{R}^n)$, and for any $T_{\mathcal{S}} \in \partial\Omega_{\mathcal{B}}$, the current $\kappa_{\#}(T_{\mathcal{S}})$ is an $(n-1)$ -chain of finite mass in \mathbb{R}^n . By Equations (4.24) and (4.25) it follows that $\kappa_{\#}(T_{\mathcal{P}}) = T_{\kappa\{\mathcal{P}\}}$ and

$\kappa_{\#}(T_S) = T_{\kappa\{S\}}$. Applying Equation (4.18), one obtains for every $T_S \in \partial\Omega_{\mathcal{B}}$ that

$$M(\kappa_{\#}(T_S)) \leq M(T_S) \left(\mathfrak{L}_{\kappa, \hat{S}} \right)^{n-1}. \quad (6.3)$$

By Equation (4.17), for every $T_{\mathcal{P}} \in \Omega_{\mathcal{B}}$,

$$N(\kappa_{\#}(T_{\mathcal{P}})) \leq N(T_{\mathcal{P}}) \sup \left\{ (\mathfrak{L}_{\kappa, \mathcal{P}})^n, (\mathfrak{L}_{\kappa, \mathcal{P}})^{n-1} \right\}. \quad (6.4)$$

For a global configuration κ , let $\kappa(\Omega_{\mathcal{B}})$ denote the collection of images of bodies under the configuration κ , *i.e.*,

$$\kappa(\Omega_{\mathcal{B}}) = \{\kappa_{\#}(T_{\mathcal{P}}) \mid T_{\mathcal{P}} \in \Omega_{\mathcal{B}}\}. \quad (6.5)$$

Similarly, the collection of surfaces at the configuration κ is

$$\kappa(\partial\Omega_{\mathcal{B}}) = \{\kappa_{\#}(T_S) \mid T_S \in \partial\Omega_{\mathcal{B}}\}. \quad (6.6)$$

A *global virtual velocity at the configuration κ* is identified with an element of the tangent space to \mathcal{Q} at κ . By Theorem 7, $\mathfrak{L}(\mathcal{B}, \mathbb{R}^n)$ is naturally isomorphic to any tangent space to \mathcal{Q} . Moreover, κ induces an isomorphism $\mathfrak{L}(\mathcal{B}, \mathbb{R}^n) \cong \mathfrak{L}(\kappa\{\mathcal{B}\}, \mathbb{R}^n)$ and an *Eulerian virtual velocity* is viewed as an element of $\mathfrak{L}(\kappa\{\mathcal{B}\}, \mathbb{R}^n)$. In what follows, we refer to $\mathfrak{L}(\kappa\{\mathcal{B}\}, \mathbb{R}^n)$ as the space of global virtual velocities at the configuration κ and use the abbreviated notation W_{κ} for it. Naturally, an element of W_{κ} may be identified with an n -tuple of sharp functions defined on $\kappa\{\mathcal{B}\}$, *i.e.*, using the Whitney topology on $\mathfrak{L}(\kappa\{\mathcal{B}\})$, $W_{\kappa} = [\mathfrak{L}(\kappa\{\mathcal{B}\})]^n$.

Focusing our attention to a particular body \mathcal{P} , one may make use of the approach of [28] and define a *virtual velocity of a body \mathcal{P} at a configuration $\kappa_{\mathcal{P}} \in \mathcal{Q}_{\mathcal{P}}$* as an element $v_{\mathcal{P}}$ in the tangent space $T_{\kappa_{\mathcal{P}}} \mathcal{Q}_{\mathcal{P}}$. It follows from Theorem 7 that one may make the identifications $T_{\kappa_{\mathcal{P}}} \mathcal{Q}_{\mathcal{P}} \cong \mathfrak{L}(\mathcal{P}, \mathbb{R}^n) \cong \mathfrak{L}(\kappa_{\mathcal{P}}\{\mathcal{P}\}, \mathbb{R}^n)$.

Theorem 10. *For every body \mathcal{P} , and every $\kappa_{\mathcal{P}} \in \mathcal{Q}_{\mathcal{P}}$, and every $\kappa \in \mathcal{Q}$ such that $\kappa|_{\mathcal{P}} = \kappa_{\mathcal{P}}$, the restriction mapping*

$$\rho_{\mathcal{P}} : T_{\kappa} \mathcal{Q} \longrightarrow T_{\kappa_{\mathcal{P}}} \mathcal{Q}_{\mathcal{P}} \quad (6.7)$$

is surjective.

Proof. We recall that Kirszbraun's theorem asserts that a Lipschitz mapping $f : A \rightarrow \mathbb{R}^m$ defined on a set $A \subset \mathbb{R}^n$ may be extended to a Lipschitz function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ having the same Lipschitz constant (see [4, Section 2.10.43] or [16, Section 6.2]). It follows immediately that any $v_{\mathcal{P}} \in \mathfrak{L}(\mathcal{P}, \mathbb{R}^n)$ may be extended to an element $v \in \mathfrak{L}(\mathcal{B}, \mathbb{R}^n)$. \square

Anticipating the properties of systems of forces to be considered below, we wish to provide the collection of restrictions of global virtual velocities to the various bodies with a finer structure than that provided by the $\|\cdot\|_{\mathfrak{L}, K}$ -semi-norms. In particular, when considering the restriction $v|_{\mathcal{P}}$ of a global virtual velocity v to a body \mathcal{P} , we wish that the magnitude of the resulting object will reflect the mass of \mathcal{P} . The *local virtual velocity for the body $T_{\mathcal{P}}$ at*

the configuration κ induced by the global virtual velocity $v \in W_\kappa$ is defined as the n -tuple of normal n -currents given by the products $v\kappa_\#(T_\mathcal{P})$ such that

$$[v\kappa_\#(T_\mathcal{P})]_i = v_i\kappa_\#(T_\mathcal{P}), \quad \text{for all } i = 1, \dots, n. \quad (6.8)$$

By Equations (5.13) and (5.11), each component $[v\kappa_\#(T_\mathcal{P})]_i$ is a normal n -current such that

$$\begin{aligned} M([v\kappa_\#(T_\mathcal{P})]_i) &\leq \sup_{y \in \kappa\{\mathcal{P}\}} |v_i(y)| M(\kappa_\#(T_\mathcal{P})), \\ &\leq \sup_{y \in \kappa\{\mathcal{P}\}} |v_i(y)| (\mathfrak{L}_{\kappa, \mathcal{P}})^n M(T_\mathcal{P}), \end{aligned} \quad (6.9)$$

and

$$\begin{aligned} N([v\kappa_\#(T_\mathcal{P})]_i) &\leq \left(\left(\sup_{y \in \kappa\{\mathcal{P}\}} |v_i(y)| \right) + n\mathfrak{L}_{v_i, \kappa\{\mathcal{P}\}} \right) N(\kappa_\#(T_\mathcal{P})), \\ &\leq \left(\left(\sup_{y \in \kappa\{\mathcal{P}\}} |v_i(y)| \right) + n\mathfrak{L}_{v_i, \kappa\{\mathcal{P}\}} \right) \\ &\quad \times \sup \left\{ (\mathfrak{L}_{\kappa, \mathcal{P}})^n, (\mathfrak{L}_{\kappa, \mathcal{P}})^{n-1} \right\} N(T_\mathcal{P}). \end{aligned} \quad (6.10)$$

In other words, the mapping $W_\kappa \times \Omega_\mathcal{B} \rightarrow \mathfrak{D}_m(\mathcal{B})$ given by $(v, T_\mathcal{P}) \mapsto v\kappa_\#(T_\mathcal{P})$ is continuous with respect to both the mass norm and the normal norm.

Similarly, the assignment of a virtual velocity $v \in W_\kappa$ to a material surface $T_\mathcal{S}$ induces an n -tuple of $(n-1)$ -chains defined by the multiplication $v\kappa_\#(T_\mathcal{S})$. Each component $[v\kappa_\#(T_\mathcal{S})]_i$ is a chain of finite mass and applying Equation (5.13), one obtains

$$\begin{aligned} M([v\kappa_\#(T_\mathcal{S})]_i) &\leq \left(\sup_{y \in \kappa\{\hat{\mathcal{S}}\}} |v_i(y)| \right) M(\kappa_\#(T_\mathcal{S})), \\ &\leq \left(\sup_{y \in \kappa\{\hat{\mathcal{S}}\}} |v_i(y)| \right) (\mathfrak{L}_{\kappa, \hat{\mathcal{S}}})^{n-1} M(T_\mathcal{S}). \end{aligned} \quad (6.11)$$

7. CAUCHY FLUXES

Alluding to the approach of [28] again, a *force on a body \mathcal{P} at the configuration $\kappa_\mathcal{P} \in \mathcal{Q}_\mathcal{P}$* is an element in the dual to the tangent space, $T_{\kappa_\mathcal{P}}^* \mathcal{Q}_\mathcal{P}$. In other words, forces on \mathcal{P} are elements of the infinite dimensional cotangent bundle $T^* \mathcal{Q}_\mathcal{P}$. For $g_\mathcal{P} \in T_{\kappa_\mathcal{P}}^* \mathcal{Q}_\mathcal{P}$, and $v_\mathcal{P} \in T_{\kappa_\mathcal{P}} \mathcal{Q}_\mathcal{P}$, the action $g_\mathcal{P}(v_\mathcal{P})$ is interpreted as the virtual power performed by the force $g_\mathcal{P}$ for the virtual velocity $v_\mathcal{P}$. It follows immediately that a force on a body \mathcal{P} at $\kappa_\mathcal{P}$ may be identified with a linear continuous functional on the space of Lipschitz mappings. Such functionals are quite irregular and will not be considered here.

Instead, we use in this section the notion of a Cauchy flux at the configuration κ , as a real valued function operating on the Cartesian product $\kappa(\partial\Omega_{\mathcal{B}}) \times W_{\kappa}$. These impose stricter conditions on the force system and resulting stress fields. The conditions to be imposed still imply that for a fixed body, a force is a continuous linear functional of the virtual velocities of that body.

A Cauchy flux represents a system of surface forces operating on the material surfaces, or more precisely, their images under κ . For a given surface and a given virtual velocity field, the value returned by the Cauchy flux mapping is interpreted as the virtual power (or virtual work) performed by the force acting on the image of the material surface under κ for the given virtual velocity.

Definition 11. A *Cauchy flux* at the configuration κ is a mapping of the form

$$\Phi_{\kappa} : \kappa(\partial\Omega_{\mathcal{B}}) \times W_{\kappa} \rightarrow \mathbb{R}, \quad (7.1)$$

such that the following hold.

Additivity: $\Phi_{\kappa}(\cdot, v)$ is additive for disjoint compatible material surfaces, *i.e.*, for every $\kappa_{\#}(T_S), \kappa_{\#}(T_{S'}) \in \kappa(\partial\Omega_{\mathcal{B}})$ compatible and disjoint,

$$\Phi_{\kappa}(\kappa_{\#}(T_{S \cup S'}), v) = \Phi_{\kappa}(\kappa_{\#}(T_S), v) + \Phi_{\kappa}(\kappa_{\#}(T_{S'}), v), \quad (7.2)$$

holds for every $v \in W_{\kappa}$.

Linearity: $\Phi_{\kappa}(\kappa_{\#}(T_S), \cdot)$ is a linear function on W_{κ} , *i.e.*, for all $\alpha, \beta \in \mathbb{R}$ and $v, v' \in W_{\kappa}$,

$$\Phi_{\kappa}(\kappa_{\#}(T_S), \alpha v + \beta v') = \alpha \Phi_{\kappa}(\kappa_{\#}(T_S), v) + \beta \Phi_{\kappa}(\kappa_{\#}(T_S), v') \quad (7.3)$$

holds for every $\kappa_{\#}(T_S) \in \kappa(\partial\Omega_{\mathcal{B}})$.

Let $v \in W_{\kappa}$ and $\kappa_{\#}(T_S) \in \kappa(\partial\Omega_{\mathcal{B}})$, then, by the linearity of the Cauchy flux,

$$\Phi_{\kappa}(\kappa_{\#}(T_S), v) = \Phi_{\kappa}\left(\kappa_{\#}(T_S), \sum_{i=1}^n v_i e_i\right) = \sum_{i=1}^n \Phi_{\kappa}(\kappa_{\#}(T_S), v_i e_i). \quad (7.4)$$

Set $\Phi_{\kappa}^i(\kappa_{\#}(T_S), u) = \Phi_{\kappa}(\kappa_{\#}(T_S), u e_i)$ for all $u \in \mathfrak{L}(\kappa\{\mathcal{B}\})$, so that Φ_{κ}^i is naturally viewed as the i -th component of the Cauchy flux at the configuration κ . One has,

$$\Phi_{\kappa}(\kappa_{\#}(T_S), v) = \sum_{i=1}^n \Phi_{\kappa}^i(\kappa_{\#}(T_S), v_i). \quad (7.5)$$

Balance: There is a number $0 < s < \infty$ such that for all components of the Cauchy flux

$$\Phi_{\kappa}^i(\kappa_{\#}(T_S), v) \leq s \|v\|_{\mathfrak{L}, \mathfrak{S}} M(\kappa(T_S)), \quad (7.6)$$

for all $\kappa_{\#}(T_S) \in \kappa(\partial\Omega_{\mathcal{B}})$ and $v \in W_{\kappa}$.

Weak balance: There is a number $0 < b < \infty$ such that for all components of the Cauchy flux

$$\Phi_{\kappa}^i(\kappa_{\#}(\partial T_{\mathcal{P}}), v) \leq b \|v\|_{\mathfrak{L}, \mathcal{P}} M(\kappa_{\#}(T_{\mathcal{P}})), \quad (7.7)$$

for all $\kappa_{\#}(T_{\mathcal{P}}) \in \kappa(\Omega_{\mathcal{B}})$ and $v \in W_{\kappa}$.

It is observed that from the balance property assumed above, for each material surface $T_{\mathcal{S}}$, $\Phi_{\kappa}(\kappa_{\#}(T_{\mathcal{S}}), \cdot)$ is continuous.

Theorem 12. *Each component of the Cauchy flux Φ_{κ} induces a unique flat $(n-1)$ -cochain in $\kappa\{\mathcal{B}\}$.*

Proof. Let σ^{n-1} be an oriented $(n-1)$ -simplex in $\kappa\{\mathcal{B}\}$. Since $\kappa\{\mathcal{B}\}$ is open there exists some n -simplex σ^n in $\kappa\{\mathcal{B}\}$ such that $\sigma^{n-1} \subset \partial\sigma^n$. Since $\kappa^{-1}\{\sigma^n\}$ is a set of finite perimeter in \mathcal{B} it follows that $\sigma^{n-1} \in \kappa(\partial\Omega_{\mathcal{B}})$. In other words, every oriented $(n-1)$ -simplex in $\kappa\{\mathcal{B}\}$ may be viewed as an element of $\kappa(\partial\Omega_{\mathcal{B}})$.

In what follows, we use extensions of Lipschitz mappings as implied by Kirszbraun's theorem. First, define a real valued function α of $(n-1)$ -simplices. Let $u : \kappa\{\mathcal{B}\} \rightarrow \mathbb{R}$ be a locally Lipschitz function in $\kappa\{\mathcal{B}\}$ such that $u(x) = 1$ for $x \in \sigma^{n-1}$, and we set

$$\alpha(\sigma^{n-1}) = \Phi_{\kappa}^i(\sigma^{n-1}, u). \quad (7.8)$$

The fact that the definition is independent of the choice of u follows from condition (7.6) and will be demonstrated below where α is extended to polyhedral $(n-1)$ -chains.

Consider a polyhedral $(n-1)$ -chain $\mathcal{A} = \sum_{j=1}^J a_j \sigma_j^{n-1}$ in $\kappa\{\mathcal{B}\}$ such that $\{\sigma_j^{n-1}\}_{j=1}^J$ are pairwise disjoint. Define the function $u : \cup_{j=1}^J \sigma_j^{n-1} \rightarrow \mathbb{R}$ by

$$u(x) = a_j \quad \text{if } x \in \sigma_j^{n-1}. \quad (7.9)$$

We now apply Kirszbraun's theorem and obtain $\tilde{u} : \kappa\{\mathcal{B}\} \rightarrow \mathbb{R}$, a Lipschitz extension to u defined on $\kappa\{\mathcal{B}\}$. By the properties postulated for Cauchy fluxes

$$\Phi_{\kappa}^i\left(\cup_{j=1}^J \sigma_j^{n-1}, \tilde{u}\right) = \sum_{j=1}^J \Phi_{\kappa}^i\left(\sigma_j^{n-1}, \tilde{u}\right) = \sum_{j=1}^J a_j \alpha\left(\sigma_j^{n-1}\right). \quad (7.10)$$

The function α is now extended to polyhedral $(n-1)$ -chains in $\kappa\{\mathcal{B}\}$ by linearity, *i.e.*,

$$\alpha(\mathcal{A}) = \alpha\left(\sum_{j=1}^J a_j \sigma_j^{n-1}\right) = \sum_{j=1}^J a_j \alpha\left(\sigma_j^{n-1}\right). \quad (7.11)$$

Thus, α is a linear functional of polyhedral $(n-1)$ -chains. The value of $\alpha(\mathcal{A})$ is independent of any particular extension of u , for given \tilde{u}' , \tilde{u} any two

Lipschitz extensions of u ,

$$\begin{aligned}
\left| \Phi_{\kappa}^i \left(\cup_{j=1}^J \sigma_j^{n-1}, \tilde{u} \right) - \Phi_{\kappa}^i \left(\cup_{j=1}^J \sigma_j^{n-1}, \tilde{u}' \right) \right| \\
= \left| \Phi_{\kappa}^i \left(\cup_{j=1}^J \sigma_j^{n-1}, \tilde{u} - \tilde{u}' \right) \right|, \\
\leq s \|\tilde{u} - \tilde{u}'\|_{\mathfrak{L}, \cup_{j=1}^J \sigma_j^{n-1}} M \left(T_{\cup_{j=1}^J \sigma_j^{n-1}} \right), \quad (7.12) \\
= 0.
\end{aligned}$$

From Equation (7.6) it follows that

$$|\alpha(\sigma^{n-1})| \leq sM(\sigma^{n-1}), \quad \text{for all } \sigma^{n-1} \in \kappa\{\mathcal{B}\}, \quad (7.13)$$

and by Equation (7.7),

$$|\alpha(\partial\sigma^n)| \leq bM(\sigma^n), \quad \text{for all } \sigma^n \in \kappa\{\mathcal{B}\}. \quad (7.14)$$

The flat norm of a the functional α is defined by

$$\begin{aligned}
F(\alpha) &= \sup \{ \alpha(\mathcal{A}) \mid \mathcal{A} \text{ is a polyhedarl } (n-1)\text{-chain}, \\
&\quad F_K(\mathcal{A}) \leq 1, K \subset \kappa\{\mathcal{B}\} \}, \quad (7.15)
\end{aligned}$$

and we obtain

$$F(\alpha) = \max \left\{ \sup_{\sigma^{n-1} \in \kappa\{\mathcal{B}\}} \frac{\alpha(\sigma^{n-1})}{M(\sigma^{n-1})}, \sup_{\sigma^n \in \kappa\{\mathcal{B}\}} \frac{\alpha(\partial\sigma^n)}{M(\sigma^{n-1})} \right\} \leq \max \{s, b\}.$$

We also recall, [4, Section 4.1.23], that polyhedral chains form a dense subspace of the space of flat chains, specifically, for every $\mathcal{A} \in F_{n-1,K}(\mathbb{R}^n)$, a compact subset $C \subset \kappa(\mathcal{B})$ whose interior contain K and $\varepsilon > 0$, there is and a polyhedral $(n-1)$ -chain $\mathcal{A}_{\varepsilon}$ supported in C such that

$$F_C(\mathcal{A} - \mathcal{A}_{\varepsilon}) \leq \varepsilon. \quad (7.16)$$

Thus, for every flat $(n-1)$ -chain \mathcal{A} we have a sequence \mathcal{A}_j such that $\lim_{j \rightarrow \infty}^F \mathcal{A}_j = \mathcal{A}$. The cochain α is uniquely extended a flat $(n-1)$ -cochain Ψ such that for every $\mathcal{A} = \lim_{j \rightarrow \infty}^F \mathcal{A}_j$

$$\Psi(\mathcal{A}) = \lim_{j \rightarrow \infty} \alpha(\mathcal{A}_j). \quad (7.17)$$

The foregoing part of the theorem is analogous to [39, Section V.4].

In order to complete the proof we need to show that for $\kappa_{\#}(T_S) \in \kappa(\partial\Omega_{\mathcal{B}})$ and $v \in \mathcal{L}_{\mathfrak{s}}(\kappa\{\mathcal{B}\})$ we obtain $\Psi(v\kappa_{\#}(T_S)) = \Phi_{\kappa}^i(\kappa_{\#}(T_S), v)$. By [4, Section 4.1.17] the class of flat chains of finite mass is the M -closure of normal currents. The chain $v\kappa_{\#}(T_S)$ is a flat $(n-1)$ -chain of finite mass. Hence, the sequence of polyhedral $(n-1)$ -chains $\{\mathcal{A}_j\}_{j=1}^{\infty}$, converging $v\kappa_{\#}(T_S)$ in the flat norm, has a convergent subsequence $\{\mathcal{A}_{j'}\}_{j'=1}^{\infty}$ such that $\{\mathcal{A}_{j'}\}$ converges to $v\kappa_{\#}(T_S)$ in the flat norm and

$$M(v\kappa_{\#}(T_S)) = \lim_{j'} M(\mathcal{A}_{j'}). \quad (7.18)$$

By the definition of α and the balance principle, Equation (7.6) the sequence $\{\alpha(\mathcal{A}_{j'})\}_{j'=1}^{\infty}$ is a Cauchy sequence in \mathbb{R} since $|\alpha(\mathcal{A}_m) - \alpha(\mathcal{A}_k)| \leq sM(\mathcal{A}_m - \mathcal{A}_k)$. Hence

$$\lim_{j' \rightarrow \infty} \alpha(\mathcal{A}_{j'}) = \Phi_{\kappa}^i(\kappa_{\#}(T_S), v). \quad (7.19)$$

Since Ψ is an extension of α it follows that $\Psi(\mathcal{A}'_j) = \alpha(\mathcal{A}'_j)$ and

$$\begin{aligned} |\Psi(v\kappa_{\#}(T_S)) - \Phi_{\kappa}^i(\kappa_{\#}(T_S), v)| &= |\Psi(v\kappa_{\#}(T_S)) - \lim_{j' \rightarrow \infty} \alpha(\mathcal{A}'_{j'})|, \\ &= |\Psi(v\kappa_{\#}(T_S)) - \lim_{j' \rightarrow \infty} \Psi(\mathcal{A}'_{j'})|, \\ &= |\Psi(v\kappa_{\#}(T_S)) - \Psi\left(\lim_{j' \rightarrow \infty} \mathcal{A}_{j'}\right)|, \\ &= |\Psi\left(v\kappa_{\#}(T_S) - \lim_{j' \rightarrow \infty} \mathcal{A}_{j'}\right)|, \\ &\leq \max\{s, b\} \lim_{j' \rightarrow \infty} F(v\kappa_{\#}(T_S) - \mathcal{A}'_{j'}) = 0, \end{aligned} \quad (7.20)$$

which completes the proof. \square

The extension of each flat $(n-1)$ -cochain from $\kappa\{\mathcal{B}\} \subset \mathbb{R}^n$ to \mathbb{R}^n is done trivially by setting its representing flat $(n-1)$ -form to vanish outside $\kappa\{\mathcal{B}\}$. We conclude that a Cauchy flux Φ_{κ} induces a unique n -tuple of flat $(n-1)$ -cochains in \mathbb{R}^n such that

$$\Phi_{\kappa}(\kappa_{\#}(T_S), v) = \sum_{i=1}^n \Psi^i(v_i \kappa_{\#}(T_S)), \quad (7.21)$$

for all $v \in W_{\kappa}$ and $\kappa_{\#}(T_S) \in \kappa(\partial\Omega_{\mathcal{B}})$. The inverse implication is provided by

Theorem 13. *An n -tuple $\{\Psi^i\}$ of flat $(n-1)$ cochains in \mathbb{R}^n induces by Equation (7.21) a unique Cauchy flux Φ_{κ} .*

Proof. For each $v \in W_{\kappa}$ and $\kappa_{\#}T_S$, the Cauchy flux $\Phi_{\kappa}(\kappa_{\#}(T_S), v)$ will be defined by Equation (7.21), and by the components

$$\Phi_{\kappa}^i(\kappa_{\#}(T_S), v_i) = \Psi^i(v_i \kappa_{\#}(T_S)). \quad (7.22)$$

The additivity (7.2) and linearity (7.3) properties clearly hold since Ψ^i is a linear function of flat $(n-1)$ -chains. For the Balance (7.6) and weak balance (7.7) properties, recall that since Ψ^i is a flat $(n-1)$ -cochain, there exists $C > 0$ such that for every flat $(n-1)$ -chain \mathcal{A} with support in K , we may

write $|\Psi^i(\mathcal{A})| \leq CF_K(\mathcal{A})$. For the balance property

$$\begin{aligned} |\Phi_\kappa^i(\kappa_\#(T_S), v_i)| &= |\Psi^i(v_i \kappa_\#(T_S))|, \\ &\leq CF_{\kappa(S)}(v_i \kappa_\#(T_S)), \\ &\leq CM(v_i \kappa_\#(T_S)), \\ &\leq C \|v_i\|_{\mathfrak{L}, \mathcal{S}} M(\kappa_\#(T_S)). \end{aligned} \tag{7.23}$$

For the weak balance

$$\begin{aligned} |\Phi_\kappa^i(\kappa_\#(\partial T_{\mathcal{P}}), v_i)| &= |\Psi^i(v_i \kappa_\#(\partial T_{\mathcal{P}}))|, \\ &\leq CF_{\kappa(\mathcal{P})}(v_i \kappa_\#(\partial T_{\mathcal{P}})), \\ &= CF_{\kappa(\mathcal{P})}(\partial(v_i \kappa_\#(T_{\mathcal{P}})) + dv \lrcorner T_{\mathcal{P}}), \\ &\leq C [F_{\kappa(\mathcal{P})}(\partial(v_i \kappa_\#(T_{\mathcal{P}}))) + F_{\kappa(\mathcal{P})}(dv \lrcorner \kappa_\#(T_{\mathcal{P}}))], \\ &\leq C [F_{\kappa(\mathcal{P})}(v_i \kappa_\#(T_{\mathcal{P}})) + F_{\kappa(\mathcal{P})}(dv \lrcorner \kappa_\#(T_{\mathcal{P}}))], \\ &\leq C [M(v_i \kappa_\#(T_{\mathcal{P}})) + M(dv \lrcorner \kappa_\#(T_{\mathcal{P}}))], \\ &\leq C \left[\sup_{x \in \kappa(\mathcal{P})} |v_i(x)| M(\kappa_\#(T_{\mathcal{P}})) + n \mathfrak{L}_{v, \kappa(\mathcal{P})} M(\kappa_\#(T_{\mathcal{P}})) \right], \\ &\leq C(n+1) \|v_i\|_{\mathfrak{L}, \kappa(\mathcal{P})} M(\kappa_\#(T_{\mathcal{P}})). \end{aligned} \tag{7.24}$$

□

Thus, Theorems 12 and 13 restate the point of view presented in [27] that the balance and weak-balance assumptions of stress theory may be replaced by the requirement that the system of forces is given in terms of an n -tuple of flat $(n-1)$ -cochains.

8. GENERALIZED BODIES AND GENERALIZED SURFACES

The representation of a Cauchy flux by an n -tuple of flat $(n-1)$ -cochains enables the generalization of the class of admissible bodies and the introduction of a larger class of material surfaces. By a generalized body we will mean a subset $\mathring{\mathcal{P}}$ of the open set \mathcal{B} such that the induced current $T_{\mathring{\mathcal{P}}}$ is a flat n -chain in \mathcal{B} . Note that the general structure constructed thus far holds for generalized bodies. For any configuration $\kappa \in \mathfrak{L}_{\text{Em}}(\mathcal{B}, \mathbb{R}^n)$, the current $\kappa_\#(T_{\mathring{\mathcal{P}}})$ is a flat n -chain in \mathbb{R}^n , and the operations $\Psi(v \kappa_\#(\partial T_{\mathring{\mathcal{P}}}))$ and $d\Psi(v \kappa_\#(T_{\mathring{\mathcal{P}}}))$ are well defined.

Definition 14. A *generalized body* is a set $\mathring{\mathcal{P}} \subset \mathcal{B}$ such that the induced current $T_{\mathring{\mathcal{P}}} = L^n \lrcorner \mathring{\mathcal{P}}$ given by

$$T_{\mathring{\mathcal{P}}}(\omega) = \int_{\mathring{\mathcal{P}}} \omega dL^n, \tag{8.1}$$

is a flat n -chain in \mathcal{B} .

By [4, Section 4.1.24] the current $T_{\mathring{\mathcal{P}}}$ is a rectifiable n -current or an *integral flat n -chain* in \mathcal{B} . Moreover, we have

$$F(T_{\mathring{\mathcal{P}}}) = M(T_{\mathring{\mathcal{P}}}) = L^n(\mathring{\mathcal{P}}). \quad (8.2)$$

The above definition of generalized bodies implies that a generalized body may be characterized as an n -rectifiable set in \mathcal{B} [4, Section 3.2.14], or alternatively, as an L^n -summable set in \mathcal{B} . The class of generalized admissible bodies is

$$\mathring{\Omega}_{\mathcal{B}} = \left\{ T_{\mathring{\mathcal{P}}} \mid \mathring{\mathcal{P}} \subset \mathcal{B}, T_{\mathring{\mathcal{P}}} \in F_n(\mathcal{B}) \right\}. \quad (8.3)$$

As mentioned in Section 3, $\mathring{\Omega}_{\mathcal{B}}$ will have the structure of a Boolean algebra if \mathcal{B} was postulated to be a bounded set. Since $N_n(\mathcal{B}) \subset F_n(\mathcal{B})$, it is clear that $\Omega_{\mathcal{B}} \subset \mathring{\Omega}_{\mathcal{B}}$. Given $T_{\mathring{\mathcal{P}}}, T_{\mathring{\mathcal{P}'}} \in \mathring{\Omega}_{\mathcal{B}}$ clearly $T_{\mathring{\mathcal{P}} \cup \mathring{\mathcal{P}'}}$ is an element of $\mathring{\Omega}_{\mathcal{B}}$. Contrary to the previous definition of bodies, a generalized body needs not be a set of finite perimeter. Although $\mathring{\mathcal{P}}$ is a bounded set, its measure theoretic boundary, $\Gamma(\mathring{\mathcal{P}})$, may be unbounded in the sense that $H^{n-1}(\Gamma(\mathring{\mathcal{P}})) = \infty$. Generally speaking, the boundary of a rectifiable set may not be a rectifiable set. A classical example of such a generalized body in \mathbb{R}^2 is the Koch snowflake. In [32], such a body is referred to as a *rough body*.

Remark 15. It is noted that although every generalized body $\mathring{\mathcal{P}}$ induces an integral flat n -chain, not every integral flat represents a generalized body. However, it seems plausible that a *flat n -class*, introduced in [42], is in one to one correspondence with the class of generalized bodies. This issue will not be considered in this work.

Considering a generalized surface, we first note that for a generalized body $T_{\mathring{\mathcal{P}}}$, $\partial T_{\mathring{\mathcal{P}}}$ is a flat $(n-1)$ -chain in \mathcal{B} . In addition, the following argument ([6, Lemma 2.1]) indicates that the restrictions of flat chains to general Borel subsets are not necessarily flat chains. Let $H_{\lambda,s}$ denote the closed half space defined by the linear functional $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$H_{\lambda,s} = \{x \in \mathbb{R}^n \mid \lambda(x) \geq s\}. \quad (8.4)$$

Let $T_{\mathring{\mathcal{P}}} \in F_{K,n}(\mathcal{B})$ be a generalized body in \mathcal{B} supported in a compact subset K of \mathcal{B} , so that $\partial T_{\mathring{\mathcal{P}}}$ is a flat $(n-1)$ -chain, and consider the chain $\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s}$. One has,

$$\begin{aligned} F_K(\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s}) &= F_K(\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s} + \partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s}) - \partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s})), \\ &\leq F_K(\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s} - \partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s})) + F_K(\partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s})), \\ &\leq F_K(\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s} - \partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s})) + F_K(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s}), \\ &\leq M(\partial T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s} - \partial(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s})) + M(T_{\mathring{\mathcal{P}}} \llcorner H_{\lambda,s}). \end{aligned} \quad (8.5)$$

Since $T_{\dot{\mathcal{P}}}$ is a chain of finite mass, $M(T_{\dot{\mathcal{P}}} \lrcorner H_{\lambda,s}) < \infty$. In addition

$$\int_{-\infty}^{\infty} M(\partial T_{\dot{\mathcal{P}}} \lrcorner H_{\lambda,s} - \partial(T_{\dot{\mathcal{P}}} \lrcorner H_{\lambda,s})) ds = M(T_{\dot{\mathcal{P}}}), \quad (8.6)$$

and so we can show that $M(\partial T_{\dot{\mathcal{P}}} \lrcorner H_{\lambda,s} - \partial(T_{\dot{\mathcal{P}}} \lrcorner H_{\lambda,s})) < \infty$ only for L^1 -almost every $s \in \mathbb{R}$.

In order to define a generalized material surface we follow [32] where the various properties of flux over fractal boundaries are investigated.

Definition 16. For a generalized body $\dot{\mathcal{P}}$, the subset $\dot{\mathcal{S}} \subset \Gamma(\dot{\mathcal{P}})$ is said to be a *trace* if there exists a set of finite perimeter M such that $\dot{\mathcal{S}} = \Gamma(\dot{\mathcal{P}}) \cap M$ and $H^{n-1}(\Gamma(\dot{\mathcal{P}}) \cap \Gamma(M)) = 0$. Each trace $\dot{\mathcal{S}}$ is associated with a unique flat $(n-1)$ -chain $T_{\dot{\mathcal{S}}}$ given by

$$T_{\dot{\mathcal{S}}} = \partial T_{\dot{\mathcal{P}} \cap M} - \partial T_M \lrcorner \dot{\mathcal{P}}. \quad (8.7)$$

For each $\omega \in \mathfrak{D}^{n-1}(\mathcal{B})$ we have

$$T_{\dot{\mathcal{S}}}(\omega) = \int_{\dot{\mathcal{P}} \cap M} d\omega(e_1 \wedge \cdots \wedge e_n) dL^n - \int_{\Gamma(M) \cap \dot{\mathcal{P}}} \omega(\vec{T}_{\partial M}) dH^{n-1}, \quad (8.8)$$

where $\vec{T}_{\partial M}$ is defined as in Equation (3.5). The set M , of finite perimeter, is referred to as the generator of the trace $\dot{\mathcal{S}}$ and it is shown in [32] that $\dot{\mathcal{S}}$ depends on M only through the intersection of $\partial T_{\dot{\mathcal{P}}}$ with M .

The collection of generalized material surfaces is defined as

$$\partial \dot{\Omega}_{\mathcal{B}} = \{T_{\dot{\mathcal{S}}} \mid \dot{\mathcal{S}} \text{ is a trace in } \mathcal{B}\}. \quad (8.9)$$

We note that by Proposition 9, for all $T_{\dot{\mathcal{S}}} \in \partial \dot{\Omega}_{\mathcal{B}}$ and $v \in W_{\kappa}$, the multiplication $v\kappa_{\#}(T_{\dot{\mathcal{S}}})$ is an n -tuple of flat $(n-1)$ -chains. Thus, by Theorem 12 the Cauchy flux is naturally extended to the Cartesian product $W_{\kappa} \times \kappa(\partial \dot{\Omega}_{\mathcal{B}})$.

9. VIRTUAL STRAINS AND THE PRINCIPLE OF VIRTUAL WORK

For $T_{\dot{\mathcal{P}}} \in \partial \dot{\Omega}_{\mathcal{B}}$ and $v \in W_{\kappa}$, $\partial(v\kappa_{\#}(T_{\dot{\mathcal{P}}}))$ is an n -tuple of flat $(n-1)$ -chains in \mathcal{B} . Thus, $\Psi(\partial(v\kappa_{\#}(T_{\dot{\mathcal{P}}}))$ is a well defined action of an n -tuple of flat $(n-1)$ -cochains on an n -tuple of flat $(n-1)$ chains. Applying Equation 5.10 for each component we obtain

$$\sum_{i=1}^n \Psi_i(\partial(v_i\kappa_{\#}(T_{\dot{\mathcal{P}}})) = \sum_{i=1}^n \Psi_i(v_i\kappa_{\#}(\partial T_{\dot{\mathcal{P}}})) - \sum_{i=1}^n \Psi_i(d\alpha_{v_i} \lrcorner \kappa_{\#}(T_{\dot{\mathcal{P}}})) . \quad (9.1)$$

Here α_{v_i} is the flat 0-chain defined in Section 5.

The terms on the right-hand side of the equation above may be interpreted as follows. The term $\sum_{i=1}^n \Psi_i(v_i\kappa_{\#}(\partial T_{\dot{\mathcal{P}}}))$ is interpreted as the virtual power performed by the surface forces for the virtual velocity v on the boundary of the body $\dot{\mathcal{P}}$. Next, for $-\Psi(\partial(v\kappa_{\#}(T_{\dot{\mathcal{P}}})) = -d\Psi(v\kappa_{\#}(T_{\dot{\mathcal{P}}}))$, the n -tuple of flat n -cochains $-d\Psi$ is viewed as the body force. Thus

the term $-d\Psi(v\kappa_{\#}(T_{\dot{\mathcal{P}}}))$ is interpreted as the virtual power performed by the body forces along the virtual velocity v on the body $\dot{\mathcal{P}}$. Finally, $\sum_{i=1}^n \Psi_i(d\alpha_{v_i} \lrcorner \kappa_{\#}(T_{\dot{\mathcal{P}}}))$ is interpreted as the virtual power performed by the Cauchy flux along the derivative of the virtual velocity v on the body $\dot{\mathcal{P}}$.

An internal virtual velocity is viewed as an element upon which the Cauchy flux will act. Thus, a generalized *internal virtual velocity* is defined as an n -tuple of flat $(n-1)$ -chains in $\kappa\{\mathcal{B}\}$. A typical internal virtual velocity will be denoted by χ and is viewed as a velocity gradient or a linear strain-like entity. Clearly, not every internal virtual velocity is derived from an external virtual velocity. Motivated by the above physical interpretation and the classical formulation of the principle of virtual work, we introduce the *kinematic interpolation map*

$$\varepsilon : \kappa(\dot{\Omega}_{\mathcal{B}}) \times W_{\kappa} \rightarrow [F_{n-1}(\kappa(\mathcal{B}))]^n \quad (9.2)$$

such that each component is given by

$$(\varepsilon(\kappa_{\#}(T_{\dot{\mathcal{P}}}), v))_i = v_i \partial \kappa_{\#}(T_{\dot{\mathcal{P}}}) - \partial(v_i \kappa_{\#}(T_{\dot{\mathcal{P}}})) . \quad (9.3)$$

Note that the map ε is disjointly additive in the first argument and linear in the second. An internal virtual velocity χ is said to be *compatible* if there are $\dot{\mathcal{P}} \in \dot{\Omega}_{\mathcal{B}}$ and $v \in W_{\kappa}$ such that

$$\chi = \varepsilon(\kappa_{\#}(T_{\dot{\mathcal{P}}}), v) . \quad (9.4)$$

Given a compatible virtual internal velocity $\chi = \varepsilon(\kappa_{\#}(T_{\dot{\mathcal{P}}}), v)$ we may write,

$$\begin{aligned} \Psi(\varepsilon(\kappa_{\#}(T_{\dot{\mathcal{P}}}), v)) &= \sum_{i=1}^n \Psi_i(v_i \partial \kappa_{\#}(T_{\dot{\mathcal{P}}}) - \partial(v_i \kappa_{\#}(T_{\dot{\mathcal{P}}})) , \\ &= \sum_{i=1}^n \Psi_i(v_i \kappa_{\#}(\partial T_{\dot{\mathcal{P}}})) - \sum_{i=1}^n d\Psi_i(v_i \kappa_{\#}(T_{\dot{\mathcal{P}}})) , \\ &= \sum_{i=1}^n d\alpha_{v_i} \wedge \Psi_i(\kappa_{\#}(T_{\dot{\mathcal{P}}})) , \end{aligned} \quad (9.5)$$

and obtain

$$\Psi(v\kappa_{\#}(\partial T_{\dot{\mathcal{P}}})) - d\Psi(v\kappa_{\#}(T_{\dot{\mathcal{P}}})) = \Psi(\varepsilon(\kappa_{\#}(T_{\dot{\mathcal{P}}}), v)) , \quad (9.6)$$

for all $T_{\dot{\mathcal{P}}} \in \dot{\Omega}_{\mathcal{B}}$ and $v \in W_{\kappa}$. We view the last equation as a generalization of the principle of virtual power.

10. STRESS

Applying the representation theorem of flat cochains, a Cauchy flux is represented by an n -tuple of flat $(n-1)$ -forms in $\kappa\{\mathcal{B}\}$. Let Ψ_i denote the flat $(n-1)$ -cochain associated with the i -th component of the Cauchy flux. Then, D_{Ψ_i} will be used to denote its representing flat $(n-1)$ -form. The

n -tuple of flat $(n-1)$ -forms in $\kappa\{\mathcal{B}\}$ representing the Cauchy flux will be denoted by D_Ψ and will be referred to as the *Cauchy stress*.

Using the representation theorem for flat forms we obtain an integral representation of the principle of virtual power given in Equation (9.6). The virtual power performed by surface forces is represented by

$$\begin{aligned}
& \sum_{i=1}^n \Psi_i (v_i \kappa_{\#} (\partial T_{\hat{\mathcal{P}}})) \\
&= \sum_{i=1}^n \left(\kappa^{\#} (d(\alpha_{v_i} \wedge \Psi_i)) \right) (T_{\hat{\mathcal{P}}}), \\
&= \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \tilde{d}(v_i D_{\Psi_i}(\kappa(x))) (D\kappa(x)(e_1) \wedge \cdots \wedge D\kappa(x)(e_n)) dL_x^n, \quad (10.1) \\
&= \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \tilde{d}(v_i D_{\Psi_i}(\kappa(x))) (e_1 \wedge \cdots \wedge e_n) J_{\kappa}(x) dL_x^n.
\end{aligned}$$

Equations (4.21) and (4.9) were used in the first and second lines. The virtual power performed by body forces is represented by

$$\begin{aligned}
& - \sum_{i=1}^n d\Psi_i (v_i \kappa_{\#} (T_{\hat{\mathcal{P}}})) \\
&= - \sum_{i=1}^n \kappa^{\#} (\alpha_{v_i} \wedge d\Psi_i) (T_{\hat{\mathcal{P}}}), \\
&= - \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \left(v_i \tilde{d}D_{\Psi_i}(\kappa(x)) \right) (D\kappa(x)(e_1) \wedge \cdots \wedge D\kappa(x)(e_n)) dL_x^n, \quad (10.2) \\
&= - \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \left(v_i \tilde{d}D_{\Psi_i}(\kappa(x)) \right) (e_1 \wedge \cdots \wedge e_n) J_{\kappa}(x) dL_x^n.
\end{aligned}$$

The virtual power performed by internal forces is represented by

$$\begin{aligned}
& \sum_{i=1}^n dv_i \wedge \Psi_i (\kappa_{\#} (T_{\hat{\mathcal{P}}})) \\
&= \sum_{i=1}^n \kappa^{\#} (d\alpha_{v_i} \wedge \Psi_i) (T_{\hat{\mathcal{P}}}), \\
&= \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \left(\tilde{d}v_i \wedge D_{\Psi_i}(\kappa(x)) \right) (D\kappa(x)(e_1) \wedge \cdots \wedge D\kappa(x)(e_n)) dL_x^n, \quad (10.3) \\
&= \sum_{i=1}^n \int_{\hat{\mathcal{P}}} \left(\tilde{d}v_i \wedge D_{\Psi_i}(\kappa(x)) \right) (e_1 \wedge \cdots \wedge e_n) J_{\kappa}(x) dL_x^n.
\end{aligned}$$

For $\kappa : \mathcal{B} \rightarrow \mathbb{R}^n$, a Lipschitz map, $\kappa^\# \Psi$ is an n -tuple of flat $(n-1)$ -cochains in \mathcal{B} . Each cochain $\kappa^\# \Psi_i$ is represented by a flat $(n-1)$ -form $D_{\kappa^\# \Psi_i} = \kappa^\# D_{\Psi_i}$. The associated n -tuple of flat $(n-1)$ -forms, $\kappa^\# D_\Psi$ is identified as the *Piola-Kirchhoff stress*

$$\left(\kappa^\# D_\Psi(x) \right)_i = J_\kappa(x) D_{\Psi_i}(\kappa(x)). \quad (10.4)$$

REFERENCES

- [1] C. Banfi and M. Fabrizio. Sul concetto di sottocorpo nella meccanica dei continui. *Rend. Acc. Naz. Lincei*, 66:136–142, 1979.
- [2] M. Degiovanni, A. Marzocchi, and A. Musesti. Cauchy fluxes associated with tensor fields having divergence measure. *Archive for Rational Mechanics and Analysis*, 147:197–223, 1999.
- [3] M. Epstein and J. Sniatycki. Fractal mechanics. *Physica D*, 220:54–68, 2006.
- [4] H. Federer. *Geometric Measure Theory*. Springer, 1969.
- [5] H. Federer and W. Fleming. Normal and integral currents. *Annals of Mathematics*, 72(3):458–520, 1960.
- [6] H. Fleming. Flat chains over a finite coefficient group. *Transactions of the American Mathematical Society*, 121:160–186, 1966.
- [7] K. Fukui and T. Nakamura. A topological property of lipschitz mappings. *Topology and its Applications*, 148:143–152, 2005.
- [8] M. Giaquinta, G. Modica, and J. Soucek. *Cartesian Currents in the Calculus of Variation I*. Springer, 1998.
- [9] E. De Giorgi. *Ennio De Giorgi: Selected Papers*. Springer, 2006.
- [10] M. E. Gurtin and L. C. Martins. Cauchy’s theorem in classical physics. *Archive for Rational Mechanics and Analysis*, 60:305–324, 1975.
- [11] M. E. Gurtin, W. O. Williams, and W. P. Ziemer. Geometric measure theory and the axioms of continuum thermodynamics. *Archive for Rational Mechanics and Analysis*, 92:1–22, 1986.
- [12] J. Harrison. Stokes’ theorem for nonsmooth chains. *Bulletin of the American Mathematical Society*, 29(2):235–242, 1993.
- [13] J. Harrison. Continuity of the integral as a function of the domain. *Journal of Geometric Analysis*, 8(5):769–795, 1998.
- [14] J. Harrison. Isomorphisms of differential forms and cochains. *Journal of Geometric Analysis*, 8(5):797–807, 1998.
- [15] J. Harrison. Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes theorems. *Journal of Physics A*, 32(28):5317–5327, 1999.
- [16] J. Heinonen. *Lectures on Analysis on Metric Spaces*. Springer, 2000.
- [17] J. Heinonen. *Lectures on Lipschitz Analysis*. University of Jyväskylä, 2005.
- [18] M.W. Hirsch. *Differential Topology*. Springer, 1976.
- [19] B. B. Mandelbrot. *The Fractal Geometry of Nature*. W.H. Freeman and Company, 1983.
- [20] A. Marzocchi and A. Musesti. The cauchy stress theorem for bodies with finite perimeter. *Rendiconti del Seminario Matematico della Università di Padova*, 109:1–11, 2003.
- [21] P. W. Michor. *Manifolds of Differentiable Mappings*. Shiva, 1980.
- [22] W. Noll. The foundations of classical mechanics in the light of recent advances in continuum mechanics. In L. Henkin, P. Suppes, and A. Tarski, editors, *The Axiomatic Method, with Special Reference to Geometry and Physics*, pages 266–281. North-Holland, 1959. Proceedings of an international symposium held at the University of California, Berkeley, December 26, 1957-January 4, 1958.

- [23] W. Noll. Lectures on the foundations of continuum mechanics and thermodynamics. *Archive for Rational Mechanics and Analysis*, 52:62–92, 1973.
- [24] W. Noll and E. G. Virga. Fit regions and functions of bounded variation. *Archive for Rational Mechanics and Analysis*, 102:1–21, 1988.
- [25] M. Ostoja-Starzewski. Continuum mechanics models of fractal porous media: integral relations and extremum principles. *Journal of Mechanics of Materials and Structures*, 4:901–912, 2009.
- [26] G. Rodnay. *Cauchy's Flux Theorem in Light of Whitney's Geometric Integration Theory*. PhD thesis, Ben-Gurion University of the Negev, 2002.
- [27] G. Rodnay and R. Segev. Cauchy's flux theorem in light of geometric integration theory. *Journal of Elasticity*, 71:183–203, 2003.
- [28] R. Segev. Forces and the existence of stress in invariant continuum mechanics. *Journal of Mathematical Physics*, 27:(1):163–170, 1986.
- [29] R. Segev and G. deBotton. On the consistency condition for force systems. *International Journal of Nonlinear Mechanics*, 26(1):47–59, 1991.
- [30] M. Silhavy. The existence of the flux vector and the divergence theorem for general cauchy flux. *Archive for Rational Mechanics and Analysis*, 90:195–212, 1985.
- [31] M. Silhavy. Cauchy's stress theorem and tensor fields with divergence in l^p . *Archive for Rational Mechanics and Analysis*, 116:223–255, 1991.
- [32] M. Silhavy. Fluxes across parts of fractal boundaries. *Milan Journal of Mathematics*, 74:1–45, 2006.
- [33] V. E. Tarsov. Continuous medium model for fractal media. *Physics Letters A*, 336:167–174, 2005.
- [34] V. E. Tarsov. Dynamics of fractal solids. *International Journal of Modern Physics B*, 19 no.27:4103–4114, 2005.
- [35] V. E. Tarsov. Fractional hydrodynamic equations for fractal media. *Annals of Physics*, 318:286–307, 2005.
- [36] V. E. Tarsov. Possible experimental test of continuous medium model for fractal media. *Physics Letters A*, 341:467–472, 2005.
- [37] C. A. Truesdell. *The Elements of Continuum Mechanics*. Springer, 1966.
- [38] C. A. Truesdell and R. Toupin. The classical field theories. In S. Flügge, editor, *Handbuch der Physik*, volume III/1. Springer, 1960.
- [39] H. Whitney. *Geometric Integration Theory*. Princeton University Press, 1957.
- [40] M. P. Wnuk and A. Yavari. Discrete fractal fracture mechanics. *Engineering Fracture Mechanics*, 75(5):1127–1142, 2008.
- [41] M. P. Wnuk and A. Yavari. A discrete cohesive model for fractal cracks. *Engineering Fracture Mechanics*, 76(4):548–559, 2009.
- [42] W. P. Ziemer. Integral currents mod 2. *Transactions of the American Mathematical Society*, Vol 5 (3):495–524, 1962.
- [43] W. P. Ziemer. Cauchy flux and sets of finite perimeter. *Archive for Rational Mechanics and Analysis*, 84:189–201, 1983.
- [44] W. P. Ziemer. *Weakly Differential Functions*. Springer, 1989.